Cooperative Game Models for Scheduling Problems

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Scope of this talk

- What I will discuss
  - A class of cooperative game models defined on scheduling problems
  - Focus on methodologies to stabilize the grand coalition when the core of a game is empty
- What I will not cover
  - A large body of work on cooperative game models related to scheduling problems
  - For example, sequencing games

Acknowledgement

My collaborators

- Prof. Lindong Liu, The University of Science and Technology, China
- Prof. Zhou Xu, The Hong Kong Polytechnic University
- Prof. Zhixin Liu, University of Michigan - Dearborn
- Dr. Liang Lu, Amazon.com

An example of cooperative game

- There are 3 players, each having a job to do
  - The cost of each working individually

    $\pi(\{1\}) = \pi(\{2\}) = \pi(\{3\}) = 10$
  - The cost of any two working collaboratively

    $\pi(\{1,2\}) = \pi(\{1,3\}) = \pi(\{2,3\}) = 14$
  - The cost of all three working collaboratively

    $\pi(\{1,2,3\}) = 18$
- Question: are the three willing to work collaboratively?
  - Sharing the cost $\pi(\{1,2,3\}) = 18$ among the players
  - A straightforward solution, (6,6,6)

How about the following ways of sharing cost?

(7,7,4)

(8,6,4)

(4,4,10)

…

The Formulation

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- We need a way of sharing the cost $\pi(\{1,2,3\}) = 18$ among the players, (x_1, x_2, x_3), satisfying

$$x_1 \le 10,$$
$$x_2 \le 10,$$
$$x_3 \le 10,$$
$$x_1 + x_2 \le 14,$$
$$x_1 + x_3 \le 14,$$
$$x_2 + x_3 \le 14,$$
$$x_1+x_2+x_3 = 18.$$

- All are feasible solutions

(6,6,6), (7,7,4), (8,6,4), (4,4,10)

They are said to be in the **core** of the game

Concepts in Cooperative Game

- A cooperative game can be depicted by (N, π)
  - N is the set of players, referred to as grand coalition
  - π: $2^N \rightarrow R$, or denoted by $\pi(S)$, is the characteristic function that specifies the cost of a coalition S (a subset of N)
- A cost allocation, $(x_1, x_2, \ldots, x_n)$, is a distribution of $\pi(N)$ to all players, i.e.,

$$\sum_{i \in N} x_i = \pi(N).$$

- An allocation $(x_1, x_2, \ldots, x_n)$ is in the **core** if for any coalition S,

$$\sum_{i \in S} x_i \leq \pi(S).$$

Any allocation in the core ensures that no player or group of players can be better off by leaving the grand coalition

The Core may be empty

- Suppose that
  $\pi(\{1\}) = \pi(\{2\}) = \pi(\{3\}) = 10$
  $\pi(\{1,2\}) = \pi(\{1,3\}) = \pi(\{2,3\}) = 14$
  $\pi(\{1,2,3\}) = 22$
- There is no feasible solution to the following constraints
  $x_1 \leq 10, x_2 \leq 10, x_3 \leq 10, x_1 + x_2 \leq 14, x_1 + x_3 \leq 14, x_2 + x_3 \leq 14$
  $x_1+x_2+x_3 = 22$
- For example, consider
  - an allocation (7,7,8), then players 2 and 3 share a cost $7+8 > \pi(\{2,3\})$
  - an allocation (8,6,8), then players 1 and 3 share a cost $8+8 > \pi(\{1,3\})$
- Can we still stabilize the grand coalition when the core is empty?

Cooperative game for single machine scheduling

- Scheduling problem: $1 \mid\mid \Sigma w_j C_j$
  - Optimal schedule is WSPT
- Game models
  - Each player has a job
  - Any coalition of players can use a machine to process their jobs

- Example with 4 jobs
  - processing times (5,6,7,8)
  - weights (4,3,2,1)
- The core is empty

Coalition	●○○○	○●○○	○○●○	○○○●	●●○○	●○●○	●○○●	○●●○	○●○●	○○●●	●●●○	●●○●	●○●●	○●●●	●●●●
Cost	20	18	14	8	53	44	33	44	32	29	89	72	64	65	115

Core and relaxed concepts

$$\mathrm{Core}(N,\pi) = \left\{\alpha : \ \alpha(N) = \pi(N),\ \alpha(S) \leq \pi(S),\ \forall S \in \mathbb{S} \setminus \{N\},\ \alpha \in \mathbb{R}^n\right\}.$$

γ-core

$$\gamma\text{-Core}(\mathrm{N},\pi) = \left\{\alpha : \ \alpha(N) = \gamma\pi(N),\ \alpha(S) \leq \pi(S),\ \forall S \in \mathbb{S} \setminus \{N\},\ \alpha \in \mathbb{R}^n\right\}.$$

the least core

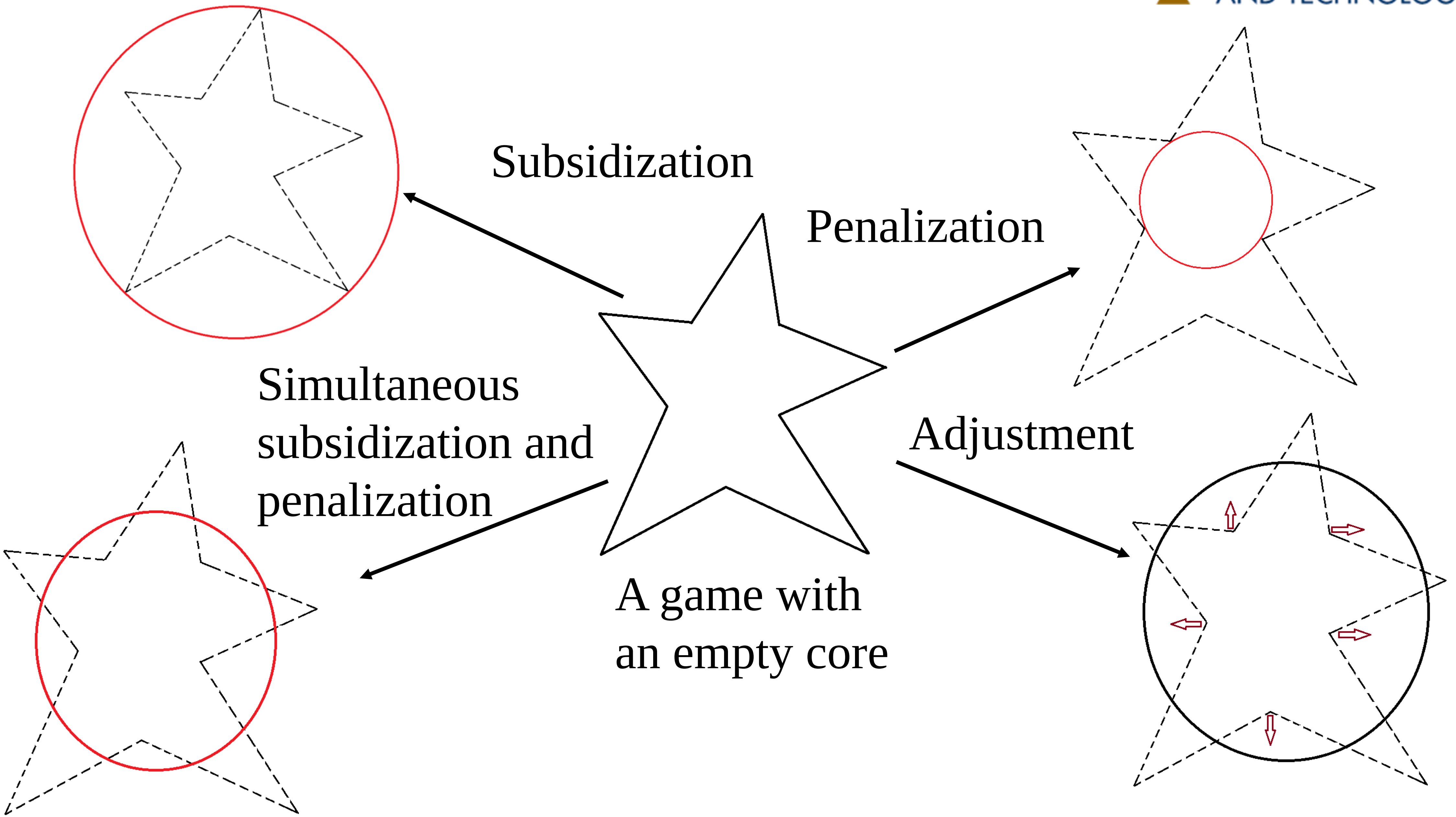
$$z^* = \min\left\{z : \alpha(N) = \pi(N),\ \alpha(S) \leq \pi(S) + z,\ \forall S \in \mathbb{S} \setminus \{N\}\right\}$$

Existing literature focuses on estimating bounds of γ and z^*

Research on cooperative games

- For a given situation
  - Define a cooperative game model
  - Check the core emptiness
  - If the core is nonempty, develop methods to find a solution in the core
  - If the core is empty, study compromised solutions such as γ-core or the least core

- What I am going to present
  - Question: For a game with empty core, is it possible to ensure the grand coalition will still be stable?
  - Basic idea: Introducing an outside party that is interested in a stable grand coalition
    - Who is this outside party? What can this outside party do?


Subsidization
Penalization
Simultaneous
subsidization and
penalization
Adjustment
A game with
an empty core

Subsidization:
Providing external resource

$$\mathrm{Core}(N,\pi) = \left\{\alpha : \ \alpha(N) = \pi(N),\ \alpha(S) \leq \pi(S),\ \forall S \in \mathbb{S} \setminus \{N\},\ \alpha \in \mathbb{R}^n\right\}.$$

Relax Budget Balanced constraint: $\alpha(N) = \pi(N)$

Minimum Subsidy to stabilize the grand coalition:

$$\omega^* = \min\left\{\pi(N) - \alpha(N) : \alpha(S) \leq \pi(S),\ \forall S \in \mathbb{S}\right\}.$$

Remarks

1. Objective function can be written as max α(N) which is also referred to as the optimal cost allocation problem.
2. The problem is equivalent to finding the γ-core
3. The difficulty: the number of constraints is exponential, and calculating each $\pi(S)$ may be NP-hard.

Subsidization: revisiting the three-player game

Recall that the grand coalition cost $\pi(\{1,2,3\}) = 22$.

The minimum subsidy is given by LP

$$\omega^* = \min 22 - (x_1+x_2+x_3)$$

Subject to

$$x_1 \le 10,$$
$$x_2 \le 10,$$
$$x_3 \le 10,$$
$$x_1 + x_2 \le 14,$$
$$x_1 + x_3 \le 14,$$
$$x_2 + x_3 \le 14.$$

The optimal solution:

$x_1 = x_2 = x_3 = 7$ and

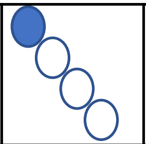
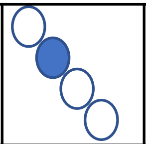
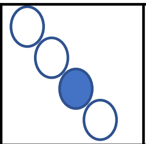
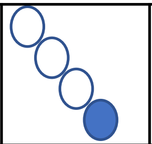
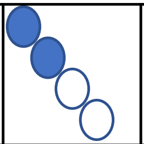
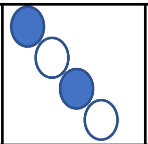
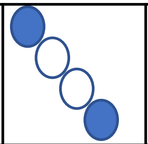
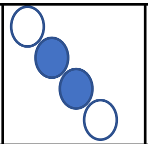
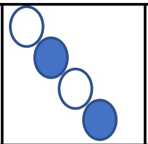
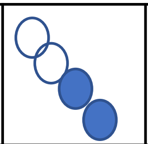
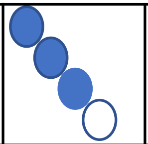
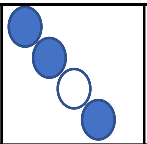
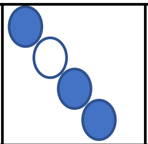
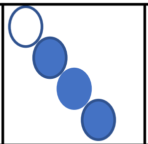
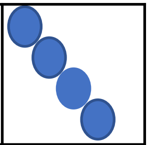
$\omega^* = 22 - 21 = 1$

Subsidization: revisiting the scheduling game

- The game of single machine scheduling $1\,|\,|\,\Sigma w_j C_j$
- Solve LP

$$\omega^* = \min 115\text{-}(\,x_1 + x_2 + x_3 + x_4)$$
$$\text{subject to } x_1 \le 20,\ x_2 \le 18,\ x_3 \le 14,\ x_4 \le 8,$$
$$x_1 + x_3 \le 14,\ x_2 + x_3 \le 14$$
$$\ldots.$$

- The optimal solution ω*=55

Coalition															
Cost	20	18	14	8	53	44	33	44	32	29	89	72	64	65	115

Penalization: imposing a surcharge on a coalition that leaves the grand coalition

$$\text{Core}(N,\pi) = \left\{\alpha : \ \alpha(N) = \pi(N), \ \alpha(S) \le \pi(S), \ \forall S \in \mathbb{S} \setminus \{N\}, \ \alpha \in \mathbb{R}^n\right\}.$$

Relax Coalition Stability constraints: $\alpha(S) \le \pi(S)$

Minimum Penalty to stabilize the grand coalition:

$$z^* = \min\left\{z : \alpha(N) = \pi(N), \ \alpha(S) \le \pi(S) + z, \ \forall S \in \mathbb{S} \setminus \{N\}\right\}.$$

Remarks
1. The problem is exactly the concept of the least core.
2. The difficulty: the number of constraints is exponential, and calculating each $\pi(S)$ may be NP-hard.

Penalization: The three-player game

- The grand coalition cost $\pi(\{1,2,3\}) = 22$.
- The minimum penalty z*

$$z^* = \min z$$

Subject to

$$x_1 \le 10+z,$$
$$x_2 \le 10+z,$$
$$x_3 \le 10+z,$$
$$x_1 + x_2 \le 14+z,$$
$$x_1 + x_3 \le 14+z,$$
$$x_2 + x_3 \le 14+z,$$
$$x_1+x_2+x_3=22$$

The optimal solution:

$x_1=x_2=x_3 = 7\frac{1}{3}$, and

$z^* = \frac{2}{3}$

Penalization: the scheduling game

- Solve LP,

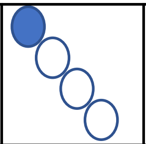
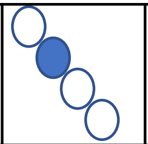
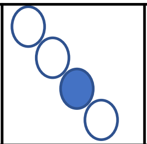
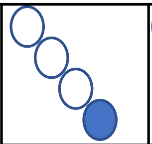
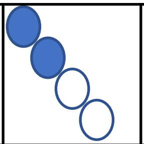
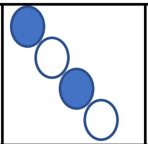
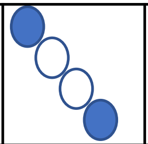
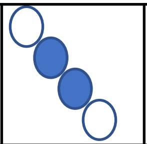
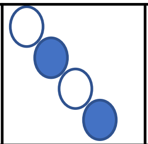
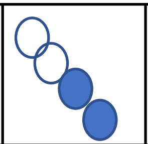
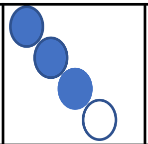
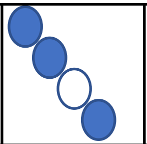
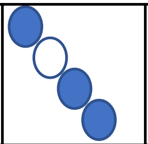
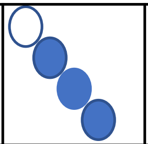
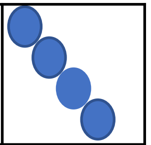
  $z^* = \min z$

  subject to $x_1 \leq 20+z$, $x_2 \leq 18+z$, $x_3 \leq 14+z$, $x_4 \leq 8+z$,

  $x_1 + x_3 \leq 14+z$, $x_2 + x_3 \leq 14+z$

  ….

- The optimal solution x_1=36.25, x_2=36.25, x_3=27.25, x_4=15.25, with z^*=19.5

Coalition															
Cost	20	18	14	8	53	44	33	44	32	29	89	72	64	65	115

Simultaneous Penalty and Subsidy

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$$\mathrm{Core}(N,\pi) = \left\{\alpha :\ \alpha(N) = \pi(N),\ \alpha(S) \le \pi(S),\ \forall S \in \mathbb{S} \setminus \{N\},\ \alpha \in \mathbb{R}^n\right\}.$$

Relax Budget Balanced constraint: $\alpha(N) = \pi(N)$

Minimum Subsidy to stabilize the grand coalition:

$$\omega^* = \min\left\{\pi(N) - \alpha(N) : \alpha(S) \le \pi(S),\ \forall S \in \mathbb{S}\right\}.$$

Relax Coalition Stability constraints: $\alpha(S) \le \pi(S)$

Minimum Penalty to stabilize the grand coalition:

$$z^* = \min\left\{z : \alpha(N) = \pi(N),\ \alpha(S) \le \pi(S) + z,\ \forall S \in \mathbb{S} \setminus \{N\}\right\}.$$

Relax Coalition Stability and Budget Balance constraints

Penalty-Subsidy Pair to stabilize the grand coalition:

$$\omega(z) = \min_{\alpha}\left\{\pi(N) - \alpha(N) :\ \alpha(S) \le \pi(S) + z,\ \forall S \in \mathbb{S} \setminus \{N\}\right\}$$

The penalty-subsidy function $\omega(z)$

- Given a specific penalty level z, we can get the minimum subsidy required to stabilize the grand coalition $\omega(z)$
  - By solving an LP with z as a parameter

- For the three-player game

$$\omega(z) = \min 22 - (x_1+x_2+x_3)$$

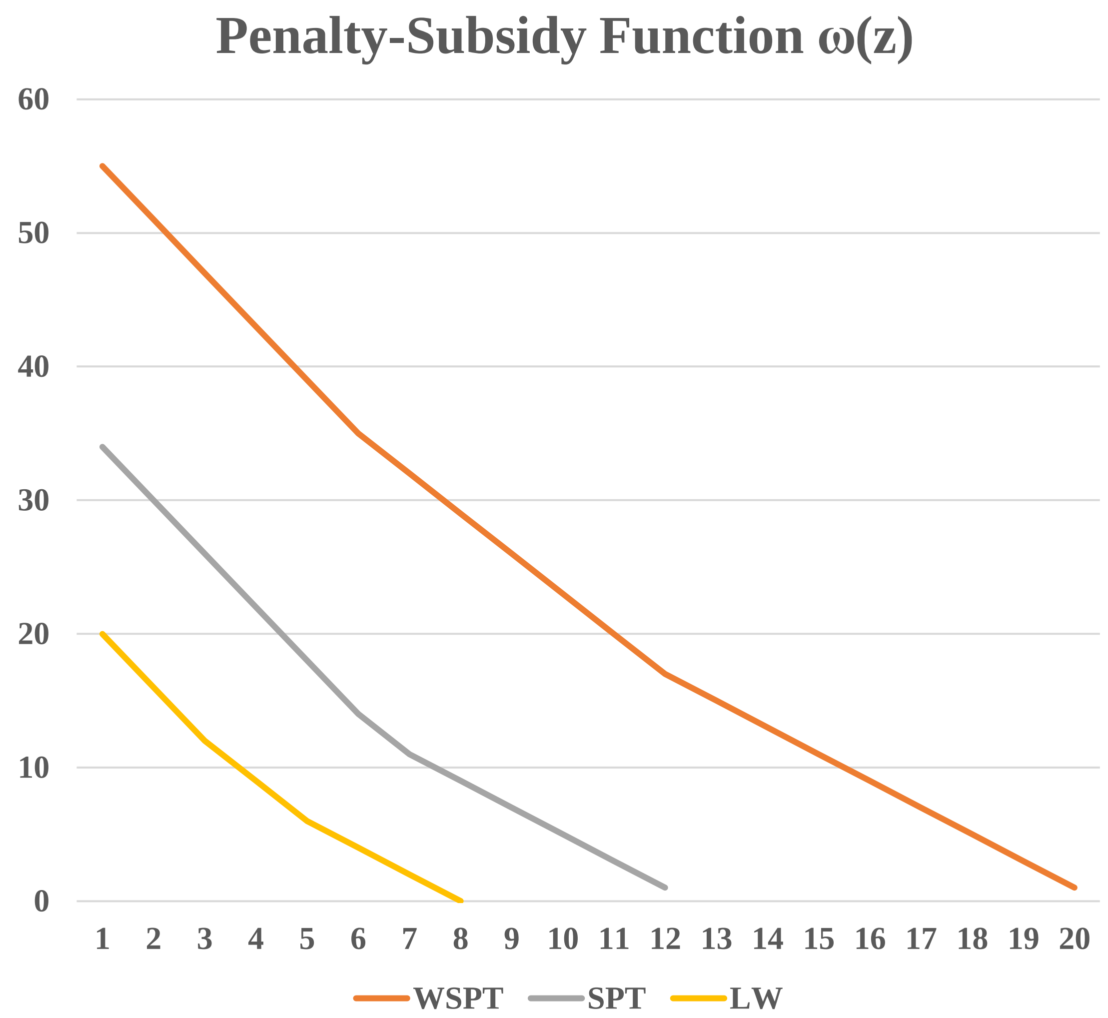
Subject to

$$x_1 \leq 10 + z,$$
$$x_2 \leq 10+z,$$
$$x_3 \leq 10+z,$$
$$x_1 + x_2 \leq 14+z,$$
$$x_1 + x_3 \leq 14+z,$$
$$x_2 + x_3 \leq 14+z$$

The optimal solution

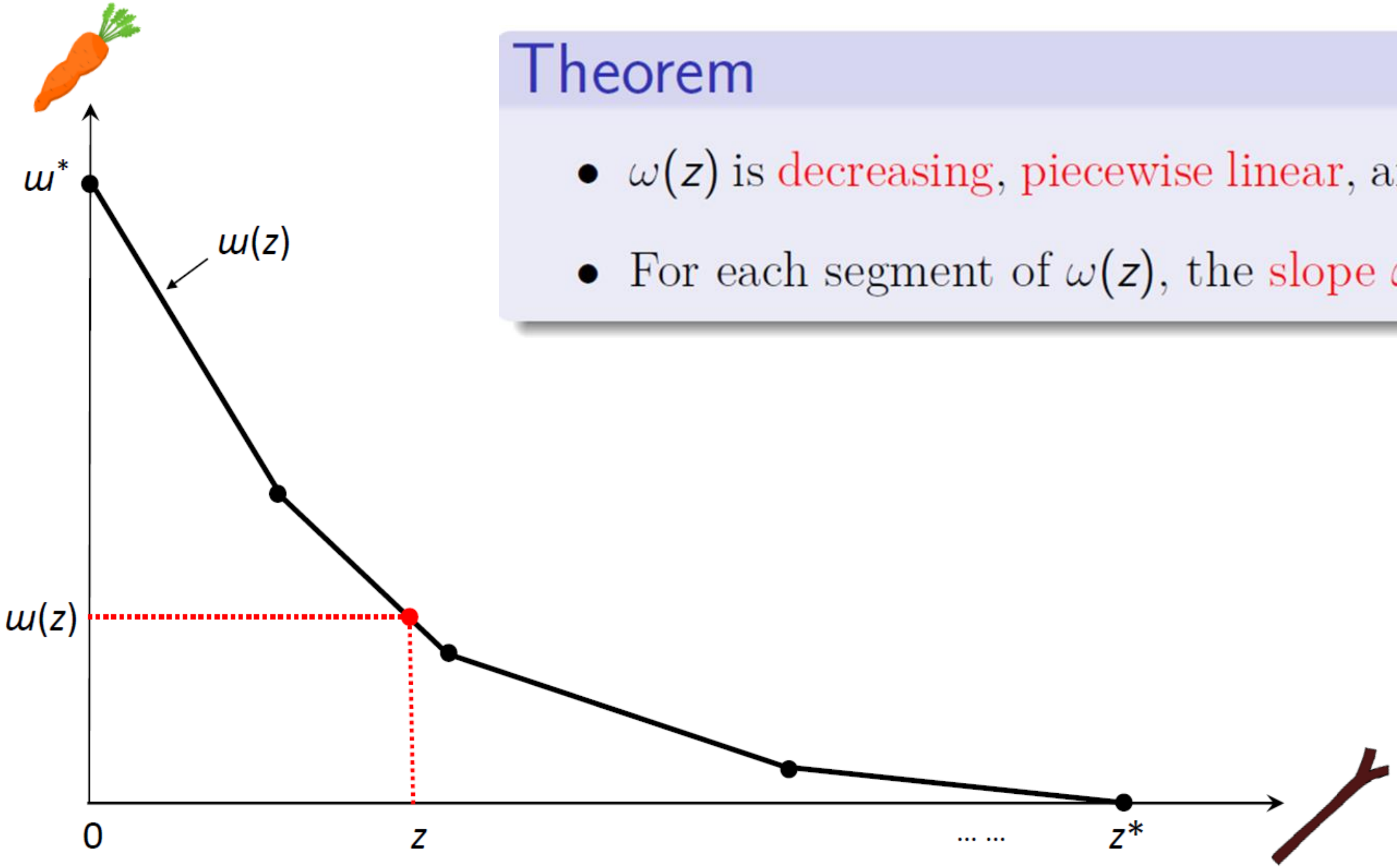
$\omega(z)$ = 1-1.5 z for $0 \leq z \leq 2/3$

Penalty-Subsidy Function for Machine Scheduling Games



- Example with 4 jobs
  - processing times (5,6,7,8)
  - weights (4,3,2,1)
- WSPT: game of $1 \mid\mid \Sigma\, w_jC_j$
- SPT: game of w_j=1
- LW: game of p_j=1

Penalty-subsidy function $\omega(z)$



Theorem

- $\omega(z)$ is decreasing, piecewise linear, and convex in $z \in [0, z^*]$.
- For each segment of $\omega(z)$, the slope $\omega'(z) \in \left[-n, -\frac{n}{n-1}\right]$.

Impact on each player

Coalition	●○○○	○●○○	○○●○	○○○●	●●○○	●○●○	●○○●	○●●○	○●○●	○○●●	●●●○	●●○●	●○●●	○●●●	●●●●
Cost	20	18	14	8	53	44	33	44	32	29	89	72	64	65	115
π(S)+z, z=5	25	23	19	13	58	49	38	49	37	34	94	77	69	70	120
α(S)	25	23	19	13	48	44	38	42	36	32	67	61	57	55	80

The maximum penalized coalition:

For a coalition S, if **α(S) =π(S)+z**, players in S face the highest penalty

Observation: for each z, any player appears in at least one of the maximum penalized coalition.

Impact on each player

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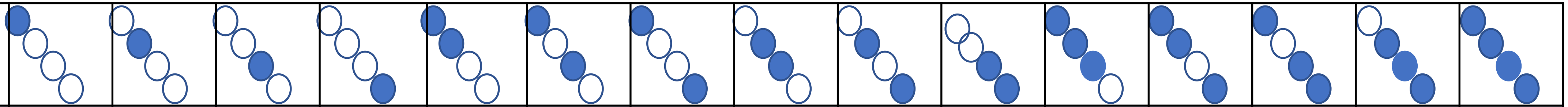
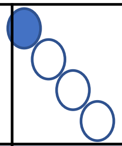
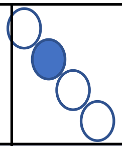
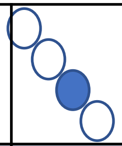
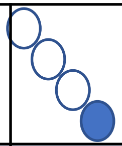
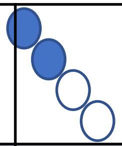
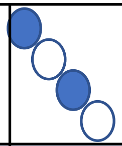
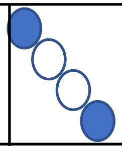
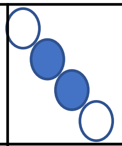
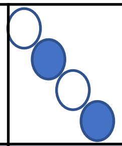
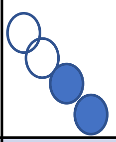
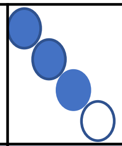
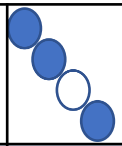
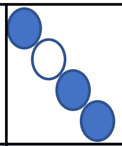
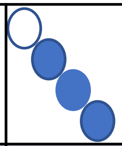
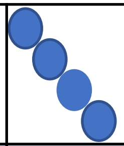


Coalition															
Cost	20	18	14	8	53	44	33	44	32	29	89	72	64	65	115
π(S)+z, z=5	25	23	19	13	58	49	38	49	37	34	94	77	69	70	120
α(S)	25	23	19	13	48	44	38	42	36	32	67	61	57	55	80
π(S)+z, z=10	30	28	24	18	63	54	43	54	42	39	99	82	74	75	125
α(S)	30	28	24	13	58	54	43	52	41	37	82	71	67	65	95

Observation: for each z, any player appears in at least one of the maximum penalized coalition.

Impact on each player

Coalition															
Cost	20	18	14	8	53	44	33	44	32	29	89	72	64	65	115
$\pi(S)+z$, z=5	25	23	19	13	58	49	38	49	37	34	94	77	69	70	120
$\alpha(S)$	25	23	19	13	48	44	38	42	36	32	67	61	57	55	80
$\pi(S)+z$, z=10	30	28	24	18	63	54	43	54	42	39	99	82	74	75	125
$\alpha(S)$	30	28	24	13	58	54	43	52	41	37	82	71	67	65	95
$\pi(S)+z$, z=19	39	37	33	27	72	63	52	63	51	48	108	91	83	84	134
$\alpha(S)$	36	36	27	15	72	63	51	63	51	42	99	87	78	78	114

Property: for each given z, any player appears in at least one of the maximum penalized coalition.

Parallel Machine Scheduling Games

- Polynomial-time solvability for ω(z) in different cases
  - Identical parallel machines, total completion time: $Pm| \mid \Sigma C_j$
  - Unrelated parallel machines, total completion time: $Qm| \mid \Sigma C_j$
  - Identical parallel machines, total weighted completion time: $Pm| \mid \Sigma w_jC_j$
  - Unrelated parallel machines, total weighted completion time: $Qm| \mid \Sigma w_jC_j$

Machines	Jobs	CP Approach	LP Approach
Identical	Unweighted	P-time	P-time
Unrelated	Unweighted	–	P-time
Identical	Weighted	Pseudo P-time (fixed m)	–
Unrelated	Weighted	Lower Bound	Upper Bound

Computing ω(z) for General Models

Integer Minimization (IM) Games:

For each coalition $S \in \mathbb{S}$, an incidence vector $y^S \in \{0,1\}^n$, with $y_j^S = 1$ if $j \in S$, and with $y_j^S = 0$ otherwise, for all $j \in N$, such that

$$\pi(S) = \min\{cx : Ax \geq By^S + E,\ x \in \mathbb{Z}^q\}.$$

- Two different approaches for IM games
  - Cutting plane method
  - LP method

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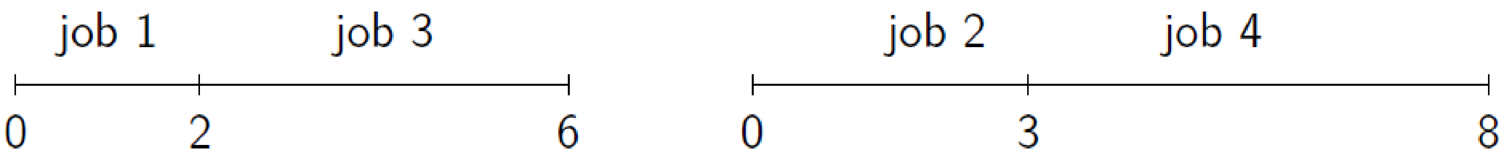
Parameters adjustment

- Parallel machine scheduling with machine activation cost
  - Each machine has an activation cost if it is used
  - Any coalition can determine the number of machines to use
  - Objective: to minimize the total completion time plus the machine activation cost
- An example
  - Processing time (2, 3, 4, 5), machine activation cost 9.5

$\pi(N) = \pi(\{1, 3\}) + \pi(\{2, 4\}) = 38$ (SPT Rule).



Coalition cost

Coalitions	Cost
$\{1\}$	11.5
$\{2\}$	12.5
$\{3\}$	13.5
$\{4\}$	14.5
$\{1,2\}$	16.5
$\{1,3\}$	17.5
$\{1,4\}$	18.5
$\{2,3\}$	19.5
$\{2,4\}$	20.5
$\{3,4\}$	22.5
$\{1,2,3\}$	25.5
$\{1,2,4\}$	26.5
$\{1,3,4\}$	28.5
$\{2,3,4\}$	31.5
$\{1,2,3,4\}$	38

Optimal Cost Allocation Problem

$$\max \ (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = 37.25 < 38$$

$$s.t. \ \ \alpha_1 \leq 11.5, \ \cdots, \ \alpha_4 \leq 14.5,$$

$$\alpha_1 + \alpha_2 \leq 16.5, \ \cdots, \ \alpha_3 + \alpha_4 \leq 22.5,$$

$$\cdots,$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 38.$$

$$\alpha^* = [6; 8.75; 10.75; 11.75]$$

Machine activation cost=9.5

Coalitions	Cost
{1}	11.5
{2}	12.5
{3}	13.5
{4}	14.5
{1, 2}	16.5
{1, 3}	17.5
{1, 4}	18.5
{2, 3}	19.5
{2, 4}	20.5
{3, 4}	22.5
{1, 2, 3}	25.5
{1, 2, 4}	26.5
{1, 3, 4}	28.5
{2, 3, 4}	31.5
{1, 2, 3, 4}	38

Machine activation cost = 10

Coalitions	Cost
{1}	12
{2}	13
{3}	14
{4}	15
{1, 2}	17
{1, 3}	18
{1, 4}	19
{2, 3}	20
{2, 4}	21
{3, 4}	23
{1, 2, 3}	26
{1, 2, 4}	27
{1, 3, 4}	29
{2, 3, 4}	32
{1, 2, 3, 4}	39

Optimal Cost Allocation Problem

$$\max \ (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = 38 < 39$$

$$s.t. \ \ \alpha_1 \le 12, \ \cdots, \ \alpha_4 \le 15,$$

$$\alpha_1 + \alpha_2 \le 17, \ \cdots, \ \alpha_3 + \alpha_4 \le 23,$$

$$\cdots,$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \le 39.$$

$$\alpha^* = [6; 9; 11; 12]$$

Subsidization funded by taxation.

1) The game still needs to be subsidized by 39-38=1.
2) Extra total machine activation cost collected is 0.5+0.5=1, just enough to subsidizes the grand coalition

Conclusion

- We have discussed cooperative games of which the core is empty.
  - Applicable to so-called Integer Minimization games
  - Including a class of scheduling problems
  - Our focus is how to stabilize the grand coalitions by using different schemes.

- Future work?
  - Lots of potentials!
  - Welcome to explore!!!

Thank you!