The marriage of Matheuristics and Scheduling

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Outline

First contact

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[2] Della Croce, F. (2016). MP or not MP: that is the question, *Journal of Scheduling*, 19:33-42.

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- MATHEuristics are not METAheuristics but are Metaheuristics,
- General definition ([1a, 1b, 1c]):
  "Matheuristic is the hybridization of mathematical programming with metaheuristics. [...] Matheuristic is not a rigid paradigm but rather a concept framework for the design of mathematically sound heuristics."

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- Interest of Matheuristics: to rely on (more and more) efficient blackbox solvers ([2]),

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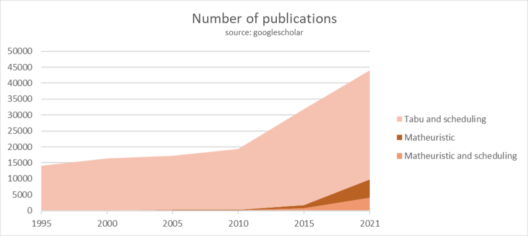
A quick look at the literature



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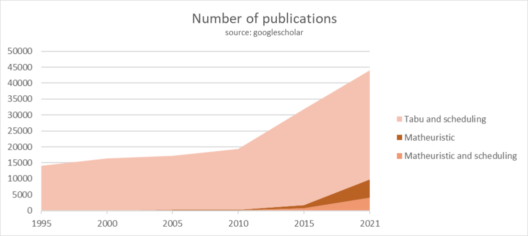

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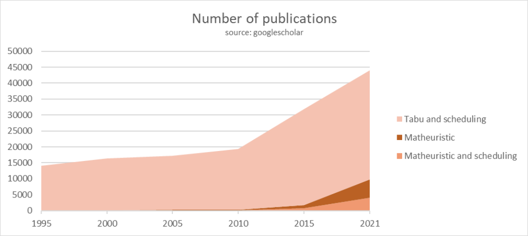

- Relatively recent,
- Mostly used for solving routing (and scheduling) problems,
- Hard to sketch a general scheme for matheuristics: RINS, Local Branching, VPLS, CMSA, Proximity Search, CRB, Relax-and-fix, POPMUSIC, ...

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- Local Search MH: use an IP in the context of a local search (requires to know an initial solution),
- Evolutionary MH: embed the solution of an IP into an evolutionary algorithm,
- This talk: a personal view based on my own experience of Local Search MH.

A general scheme (Local Search MH)

- Matheuristic as *LNS* heuristics ([1,3]),

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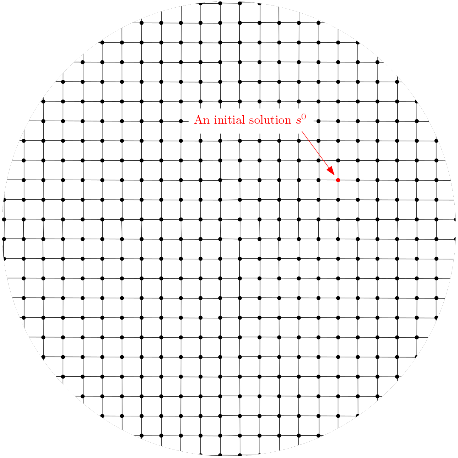

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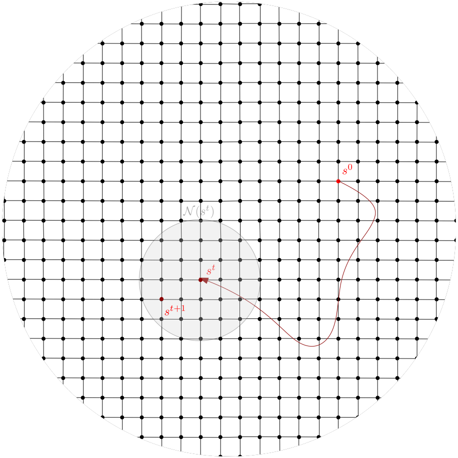

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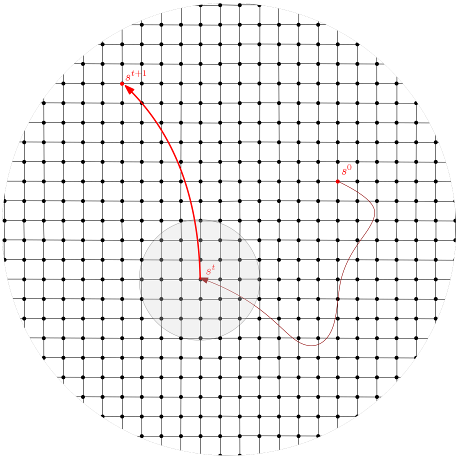
- In case of local optimum: *diversification* by MIP.



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Outline

A general scheme (intensification)

- Consider a MIP formulation of your problem (**crucial choice**),

$$\min \sum_{j=1}^{n} C_{[j]} \tag{1}$$

subject to

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \forall j = 1, \ldots, n \tag{2}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i = 1, \ldots, n \tag{3}$$

$$C_{[1]} = \sum_{i=1}^{n} (p_i + r_i) x_{i1} \tag{4}$$

$$C_{[j]} \geq C_{[j-1]} + \sum_{i=1}^{n} p_i x_{ij} \qquad \forall j = 2, \ldots, n \tag{5}$$

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$$x_{ij} \in \{0, 1\}, \quad C_{[j]} \geq 0 \tag{7}$$

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$$\mathbf{x_{ij} = x^t_{ij}} \qquad \forall \mathbf{x_{ij}} \in \mathcal{S}^\mathbf{t} \quad (8)$$

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- Variable-fixing based intensification: VPLS, Relaxation-Induced Neighbourhood Search (RINS), Fix-and-Optimize, ...
- Distance based intensification: local branching,
  1. Determine a subset $\mathcal{S}^t$ of variables x_{ij},
  2. Add a "distance measure" constraint, *e.g.* the Hamming distance:

$$\min \sum_{j=1}^{n} C_{[j]}$$

subject to

(1-7)

$$\Delta_{\mathcal{S}^t}(x, x^t) = \sum_{(ij)\in\mathcal{S}^t, x_{ij}^t=0} x_{ij} + \sum_{(ij)\in\mathcal{S}^t, x_{ij}^t=1} (1 - x_{ij}) \leq k$$

with k a given parameter.

VPLS: on which problem?

- We illustrate the Variable Partitioning Local Search (VPLS) on the $F2||\sum_j C_j$ problem ([4]),

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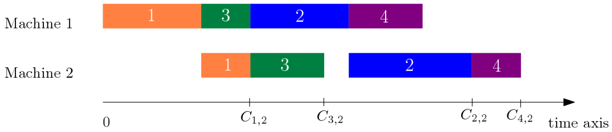

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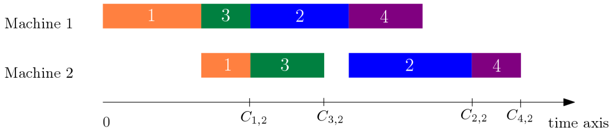

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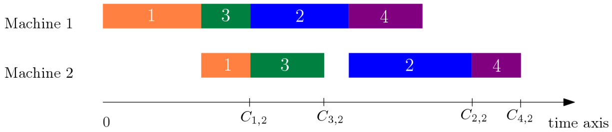

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- `Neighbourhood definition` $\mathcal{N}(s^t)$:

h=12

$$s^t = (1, 15, 3, 4, 2, 12, 8, 10, 6, 13, 9, 7, 11, 5, 14)$$

$$s^{t+1} = (1, 15, 3, 8, 4, 12, 2, 10, 6, 13, 9, 7, 11, 5, 14)$$

r (random)

$$\mathcal{S}^t = \{x_{ij} | i = 1..r-1, r+h+1, ...n, j = 1..n\}$$

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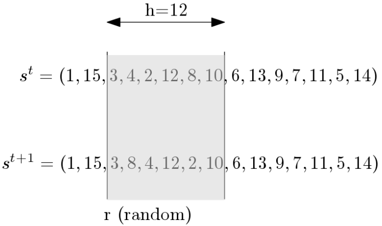

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$\Rightarrow$ well suited for permutation problems.

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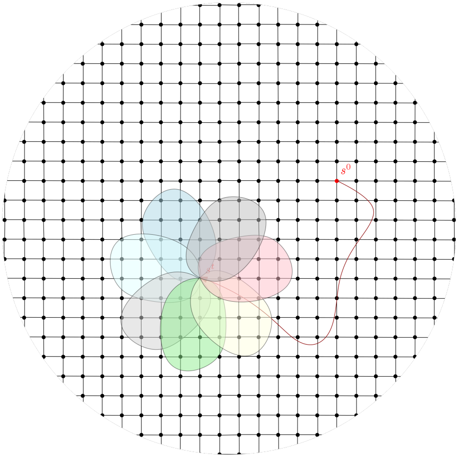

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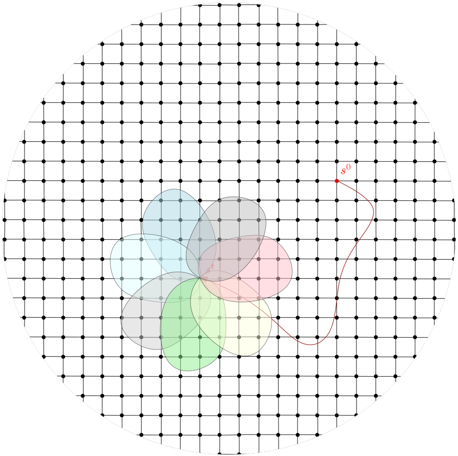

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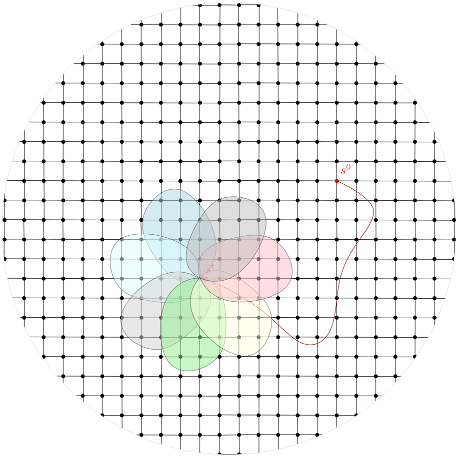
- `Stopping condition:` a given time limit T_{stop} is reached or no improving neighbourhood.



VPLS: the cake

- Experimental results on randomly generated instances ([4]):

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- Experimental results on randomly generated instances ([4]):
  - The choice of $h = 12$ is a good trade-off between time spent at each intensification phase and quality of the computed solution,
  - Results on instances with $n = 100$ ($T_{stop} = 60s$),

VPLS/LB (%)	CPLEX$_t$/LB (%)	VPLS/CPLEX$_t$ (%)	CPLEX$_t$-VPLS
0.26	0.46	0.20	370

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VPLS/LB (%)	$\texttt{CPLEX}_t$/LB (%)	VPLS/$\texttt{CPLEX}_t$ (%)	$\texttt{CPLEX}_t$-VPLS
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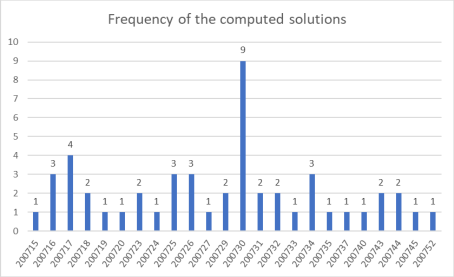

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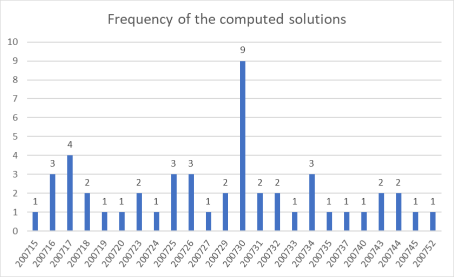

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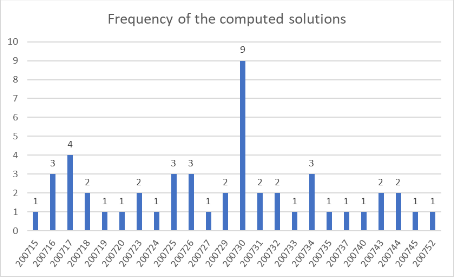

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- Considering windows of positions makes sense for permutation problems,
- Can be extended to problems with assignment... but is it the best choice?

VPLS: Conclusions

- Use of distance based neighbourhood (case of the Hamming distance),

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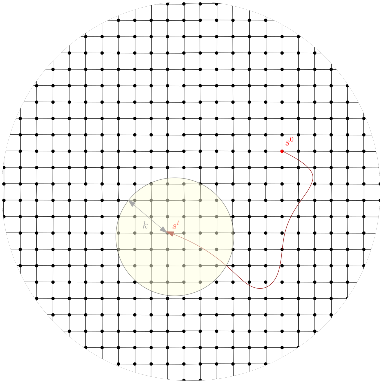

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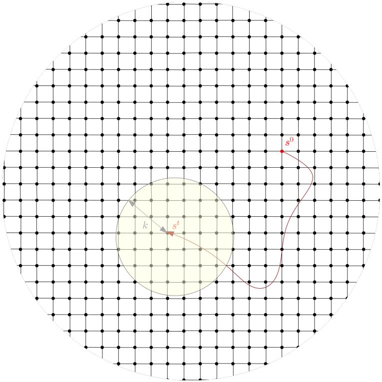

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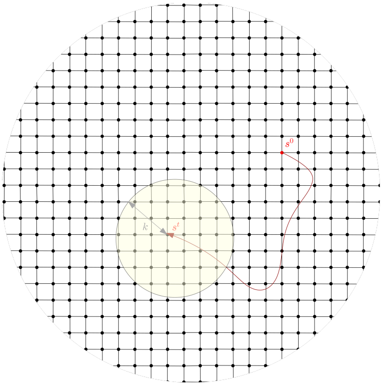

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Outline

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- Local Branching is a perfect example of a matheuristic using both *intensification* and *diversification*.

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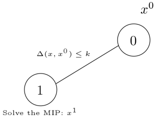

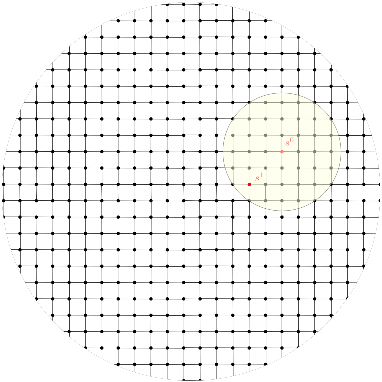

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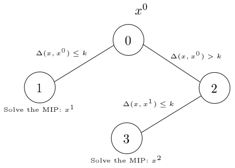

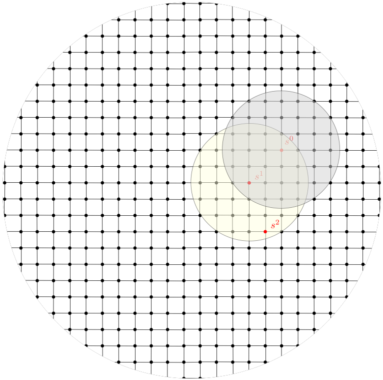

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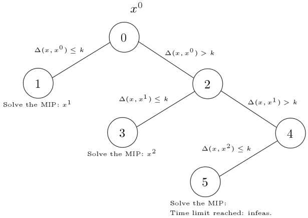

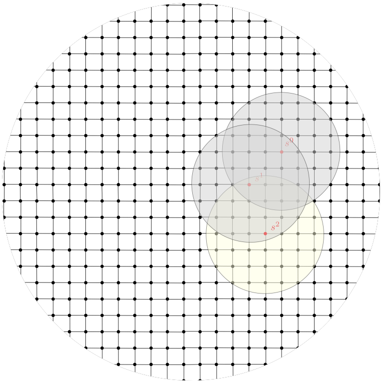

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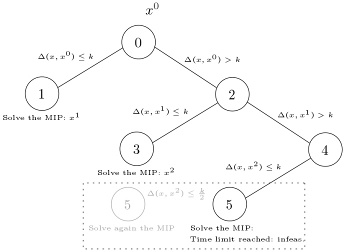

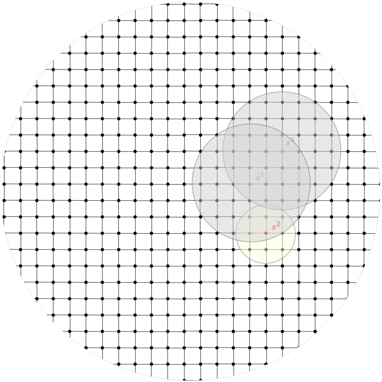

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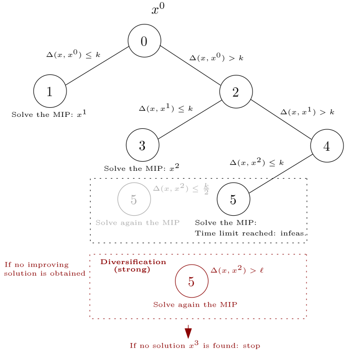

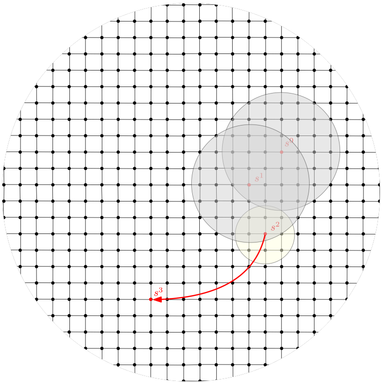

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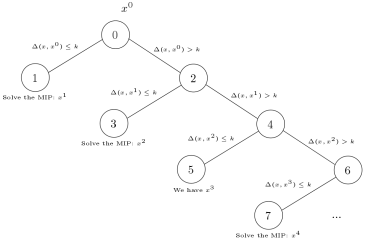

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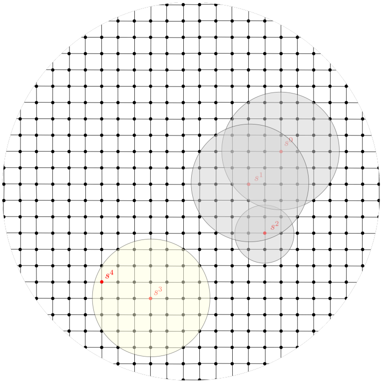

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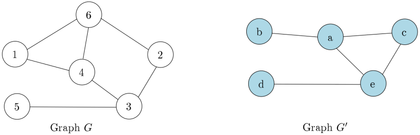

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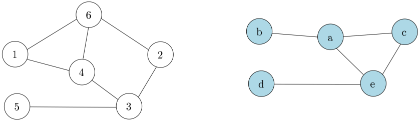

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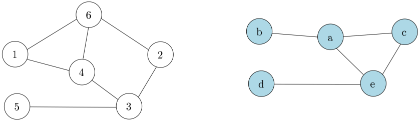

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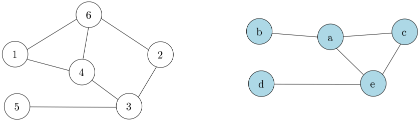

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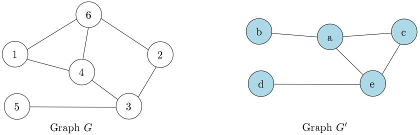

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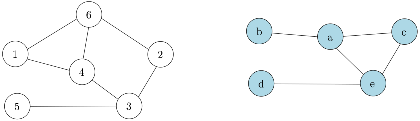

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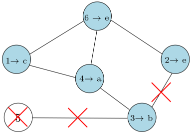

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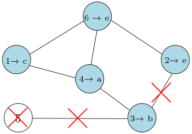

A matching

... and we want to know how much G resembles to G'.
  - In addition to the graphs, we know:
    1. The cost for deleting or creating a vertex from G: a constant τ,
    2. The cost for deleting or creating an edge in G: the same constant τ,
    3. The cost c_{jk} for matching $j \in V$ with $k \in V'$.
  - Find the matching of G and G' which minimizes the total matching cost,
  - This problem is strongly $\mathcal{NP}$-hard.

[10] Darwiche, M., Conte, D., Raveaux, R., T'kindt, V. (2019). A local branching heuristic for solving a Graph Edit Distance problem, *Computers and Operations Research*, 106:225–235.

Local Branching on the GED: the ingredients

- This is an assignment problem for which we use the following IP formulation ([11]),

$$\text{Minimize} \sum_{i=1}^{N}\sum_{j=1}^{N}\left(c_{ij}x_{ij} + \frac{\tau}{2}(s_{ij} + t_{ij})\right)$$

st

$$\sum_{k=1}^{N} A_{ik}x_{kj} - \sum_{c=1}^{N} x_{ic}A'_{cj} + s_{ij} - t_{ij} = 0 \qquad \forall i, j = 1..N$$

$$\sum_{i=1}^{N} x_{ik} = \sum_{j=1}^{N} x_{kj} = 1 \qquad \forall k = 1..N$$

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  6. To diversify with solve the IP with the constraint:
  $$\Delta_{\mathcal{S}_I^t}(x, x^t) \geq \ell.$$

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  - MUTA: 80 graphs from 10 to 70 vertices (6400 instances).
    - $k = 20$, $\ell = 30$,
    - $T_{node} = 180s$, $T_{solve} = 900s$, $Div_{solve} = 3$.

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[1] Computed by solving the IP formulation with a time limit of 10h per instance

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  - Average CPU time: 750s on the largest instances,
  - Gap to the best known solution[1]: $< 0.78\%$.

$\Rightarrow$ Outperforms all the known heuristics (in 2021) on the GED problem.

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  5. Diversification seems to be really useful.

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  3. Adjust the neighbourhood size dynamically.

Outline

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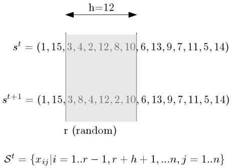

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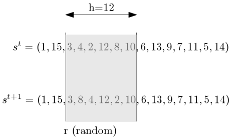

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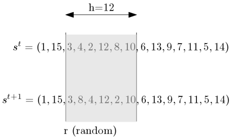

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- The neighbourhoods to explore are defined by r and h,
- Can we use Machine Learning to predict the best r and h for a given s^t?

The ml-VPLS heuristic

- Ideal goal: to have an oracle (predictor) capable of predicting the values of r and h for a given s^t,

[40] T'kindt, V., Raveaux, R. (2022). A learning based matheuristic to solve the two machine flowshop scheduling problem with sum of completion times. *23rd French Conference on Operations Research and Decision Aid (ROADEF)*, Lyon, France.

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- Use of structured machine learning to solve this classification problem (features based approach, [40]),

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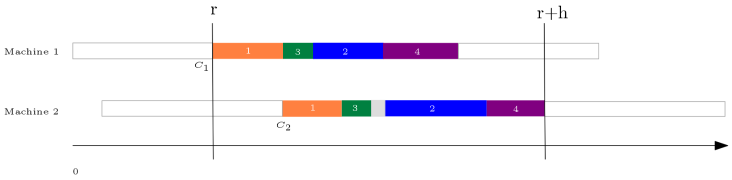
- Predictor $p()$ is a neural network and the θ are weights (Deep Learning).

The ml-VPLS heuristic

- A set of 90 features,



Descriptive features:

- C_1, C_2, $\sum_{j=r}^{r+h} p_{s[j],1}$, $\sum_{j=r}^{r+h} p_{s[j],2}$,
- In $[r; r+h]$: ratios $\frac{p_{j,1}}{p_{j,2}}$, idle times on M_2, number of jobs not in SPT order on M_2, ...
- In $[r+h+1; n]$: idle times on M_2.

Informative features:

- Upper bound on the gain (on $\sum_{j=r+h+1}^{n} C_j$) in rescheduling $[r; r+h]$,
- Lower bounds on the gain (on $\sum_{j=r}^{r+h} C_j$) in rescheduling $[r; r+h]$,
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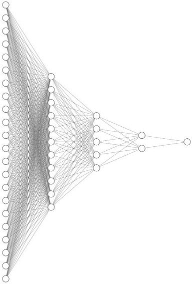
- Features are normalized and standardized.

The ml-VPLS heuristic

- Predictor (p) is a fully connected neural network:
  - It operates in a vector space ($\in \mathbb{R}^{90}$).
  - Fast inference (prediction time).
  - Other models were put to the test such as 1-dimensional CNNs but inference was too slow.
  - Number of parameters : 14 0000
  - Number of layers : 7
  - Overfitting breakers : Dropout, L1 regularization.



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- To generate the *training*, *validation* and *test* databases, the same protocol has been used:

	Train	Validation	Test
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  2. Run MATH in which all windows $[r; r+h]$ are tested. For each $x = [r; h; s]$ record $\phi(x)$ and the result $y = 1/0$,

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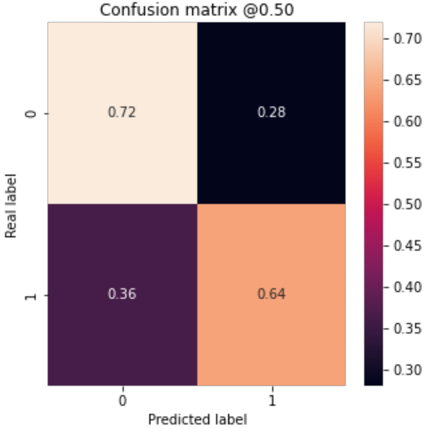
The `ml-VPLS` heuristic: After training



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  - `ml-VPLS+`: optimize the 70% best intervals (predictions) at each iteration.

Efficiency of `ml-VPLS`

- We randomly generate 50 instances per problem size and we compare four versions of the matheuristics:
  - `VPLS`: the original matheuristic,
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- On each instance, VPLS, r-VPLS and ml-VPLS are ran 10 times and the average solution value is used to compute statistics,
- A total time limit of 60s per instance for VPLS, r-VPLS and ml-VPLS.

Efficiency of ml-VPLS

	$\delta_{avg}(\%)$	$\delta_{max}(\%)$	$T_{avg}(s)$	$T_{max}(s)$	$T2best_{avg}(s)$	$T2best_{max}(s)$
VPLS	0.0031	0.046	61.13	61.36	5.62	22.18
r-VPLS	0.0034	0.060	61.14	61.39	5.88	24.58
ml-VPLS	0.0187	0.083	61.13	61.43	2.55	14.24
ml-VPLS+	0.0055	0.048	7.38	22.87	3.36	15.13

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- Results for $n = 50$ jobs -

- The trained predictor generalizes well for $n > 50$,
- Machine Learning seems interesting to make VPLS converging faster.

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- Not enough efficient in the above example but improvement is on-going!
- What is a "good" neighbourhood?
- We can also imagine other possible use of Machine Learning: selection of variables (set $\mathcal{S}^t$), value of parameters (like k and ℓ in local branching), ...

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[9] [14]	[9] [12] [31] [35]	[4] [13] [16] [21] [34]	[18]	[19] [25] [27] [29] [30]

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- Matheuristics can be also *constructive heuristics* or can result from the hybridization of *evolutionary algorithms* and MIP....

Constructive MH	Evol. Alg. MH	Others
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$\Rightarrow$ Recommendation of the day: if you have a MIP, set up a matheuristic!

Thank you for your attention!

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[4] Della Croce, F., A. Grosso, F. Salassa (2014). A matheuristic approach for the two-machine completion time flow shop problem, *Annals of Operations Research*, 213:67–78.
[9] Yang, F., Roel, L. (2021). Scheduling hybrid flow shops with time windows, *Journal of Heuristics*, 27:133–158.
[12] Della Croce, F., Salassa, F. (2014). A variable neighborhood search based matheuristic for nurse rostering problems, *Annals of Operations Research*, 218:185–199.
[13] Della Croce, F., Salassa, F., T'kindt, V. (2014). A hybrid heuristic approach for single machine scheduling with release times, *Computers and Operations Research*, 45:7–11.
[14] Smet, P., Wauters, T., Mihaylov, M., Vanden Berghe, G. (2104). This shift minimisation personnel task scheduling problem: A new hybrid approach and computational insights, *Omega*, 46:64-73.
[15] Lin, S.-W., Ying, K.-C. (2016). Optimization of makespan for no-wait flowshop scheduling problems using efficient matheuristics, *Omega*, 64:115-125.
[16]. Deghdak, K., T'kindt, V., Bouquard, J.-L. (2016). Scheduling evacuation operations, *Journal of Scheduling*, 19:467-478.
[17]. Fanjul-Peyro, L., Perea, F. Ruiz, R. (2017). Models and matheuristics for the unrelated parallel machine scheduling problem with additional resources, *European Journal of Operational Research*, 260:482-493.
[18] Doi, T., Nishi, T., Voss, S. (2018). Two-level decomposition-based matheuristic for airline crew rostering problems with fair working time, *European Journal of Operational Research*, 267:428-438.
[19] Lindahl, M., Sorensen, M., Stidsen, T.R. (2018). A fix-and-optimize matheuristic for university timetabling, *Journal of Heuristics*, 24:645:665.
[20] Monch, L., Roob, S. (2018). A matheuristic framework for batch machine scheduling problems with incompatible job families and regular sum objective, *Applied Soft Computing*, 68:835-846.

Thank you for your attention!

[21] Ta, Q.C., Billaut, J.-C., Bouquard, J.-L. (2018). Matheuristic algorithms for minimizing total tardiness in the m-machine flow-shop scheduling problem, *Journal of Intelligent Manufacturing*, 29:617-628.
[22] Woo, Y.-B., Kim, S. (2018). Matheuristic approaches for parallel machine scheduling problem with time-dependent deterioration and multiple rate-modifying activities, *Computers and Operations Research*, 95:97-112.
[23] Meisel, F., Fagerholt, K. (2019). Scheduling two-way ship traffic for the Kiel Canal: Model, extensions and a matheuristic, *Computers and Operations Research*, 106:119-132.
[24] Ozer, E. A., Sarac, T. (2019). MIP models and a matheuristic algorithm for an identical parallel machine scheduling problem under multiple copies of shared resources constraints, *TOP*, 27:94-124.
[25] Guimaraes, L., Klabjan, D., Almada-Lobo, B. (2013). Pricing, Relaxing and fixing under lot sizing and scheduling, *European Journal of Operational Research*, 230:399-411.
[26] Ferreira, D., Morabito, R., Rangel, S. (2009). Solution approaches for the soft drink integrated production lot-sizing and scheduling problem, *European Journal of Operational Research*, 196:697-706.
[27] James, R.J.W, and Almada-Lobo, B. (2011). Single and parallel machine capacitated lotsizing and scheduling: new iterative MIP-based neighborhood search heuristics, *Computers and Operations Research*, 38:1816-1825.
[28] Kang, S., Malik, K., Thomas, L.J. (1999). Lotsizing and Scheduling on Parallel Machines with Sequence-Dependent Setup Costs, *Management Science*, 45(2):131-295.
[29] Goerler, A., Lalla-Ruiz, E., Voss, S. (2020). Late acceptance Hill-Climbing matheuristic for the general lot sizing and scheduling problem with rich constraints, *Algorithms*, 13(138):1-26.
[30] Thiruvady, D., Blum, C., Ernst, A.T. (2020). Solution merging in matheuristics for resource constrained job scheduling, *Algorithms*, 13(256):1-31.
[31] Ahmadian, M.M., and Salehipour, A. (2021). The just-in-time job-shop scheduling problem with distinct due-dates for operations, *Journal of Heuristics*, 27:175-204.
[32] Chandrasekharan, R. C., Smet, P., Wauters, T. (2021). An automatic constructive matheuristic for the shift minimization personnel task scheduling problem, *Journal of Heuristics*, 27:205-227.
[33] Dang, Q.-V., van Diessen, T., Martagan, T., Adan, I. (2021). A matheuristic for parallel machine scheduling with tool replacement, *European Journal of Operational Research*, 291:640-660.
[34] Della Croce, F., Grosso, A., Salassa, F. (2021). Minimizing total completion time in the two-machine no-idle no-wait flow shop problem, *Journal of Heuristics*, 27:159-173.
[35] Dupin, N., Talbi, E.-G. (2021). Matheuristics to optimize refueling and maintenance planning of nuclear power plants, *Journal of Heuristics*, 27:63-105.

Thank you for your attention!

[36] Guzman, E., Andres, B., Poler, R. (2022). Matheuristic algorithms for Job-Shop scheduling problem using a disjunctive mathematical model, *Computers*, 11(1).
[37] Singh, N., Dang, Q.-V., Akcay, A., Adan, I., Martagan, T. (2022). A matheuristic for AGV scheduling with battery constraints, *European Journal of Operational Research*, 298:855-873.
[38] Hong, J., Moon, K., Lee, K., Lee, K., Pinedo, M.L. (2022). An iterated greedy matheuristic for scheduling in steelmaking-continuous casting process, *International Journal of Production Research*, 60(2):623-643.
[39] Tarhan, I., Oguz, C. (2022). A matheuristic for the generalized order acceptance and scheduling problem, *European Journal of Operational Research*, 299(1):87-103.