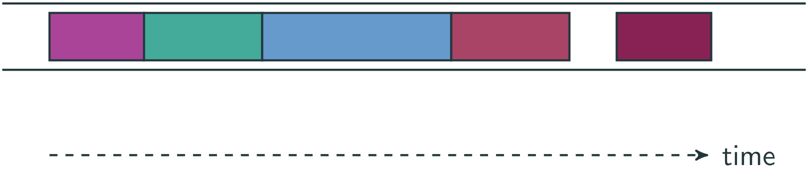
Recent Advances in Flow Time Scheduling

Lars Rohwedder

Maastricht University

Scheduling Seminar

flow time = time between arrival/release and completion



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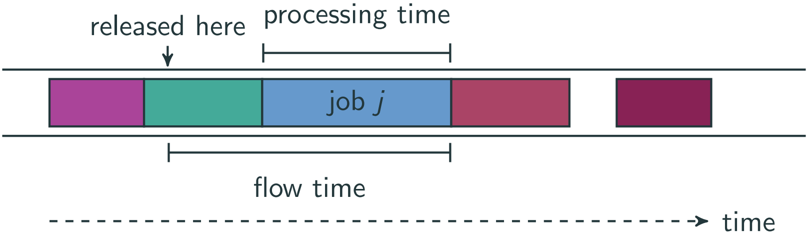
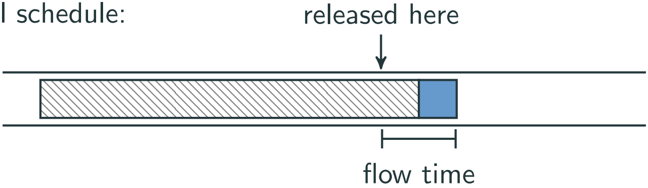


A few reasons why you should be interested:

- natural and classic measure of quality-of-service in scheduling
- related to completion time minimization
  (completion time = flow time if job released at zero)
- intuitively should be much harder than optimizing completion time, but there are surprises

Why it seems harder

optimal schedule:



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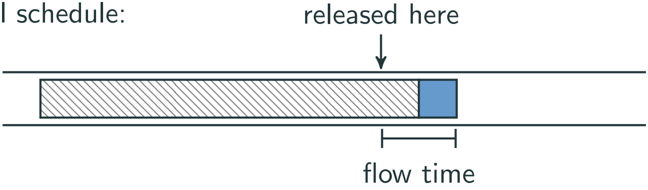
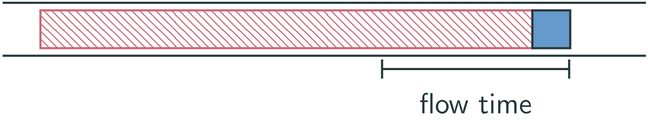


slightly suboptimal schedule:



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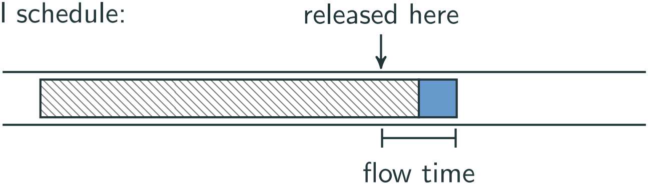

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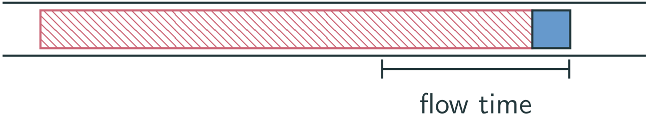

$\rightsquigarrow$ breaks many known techniques

Three concrete examples

Fundamental problems in flow time scheduling

$1 \mid \mathrm{pmpt}, r_j \mid \sum_j w_j F_j$	NP-hard,	$\leq O(1)$	
$P \mid r_j \mid F_{\max}$	NP-hard,	≤ 3	(state: 3 years ago)
$R \mid r_j \mid F_{\max}$	≥ 1.5,	$\leq O(\log n)$	

Related problems considering completion time

$1 \mid \mathrm{pmpt}, r_j \mid \sum_j w_j C_j$	NP-hard,	PTAS
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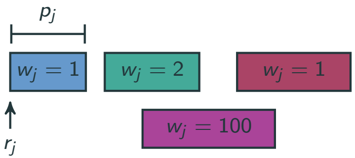
* non-constructive integrality gap

$1 \mid \mathrm{pmpt}, r_j \mid \sum_j w_j F_j$

jobs:



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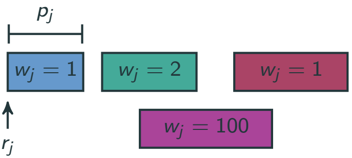

schedule:

$1 \mid \mathrm{pmpt}, r_j \mid \sum_j w_j F_j$

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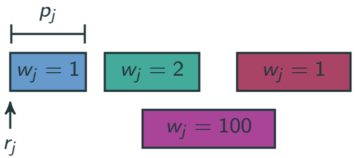

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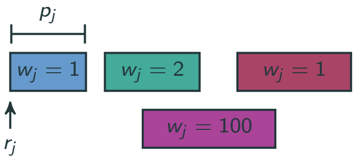

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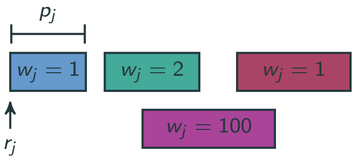

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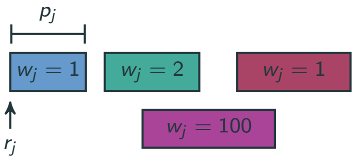

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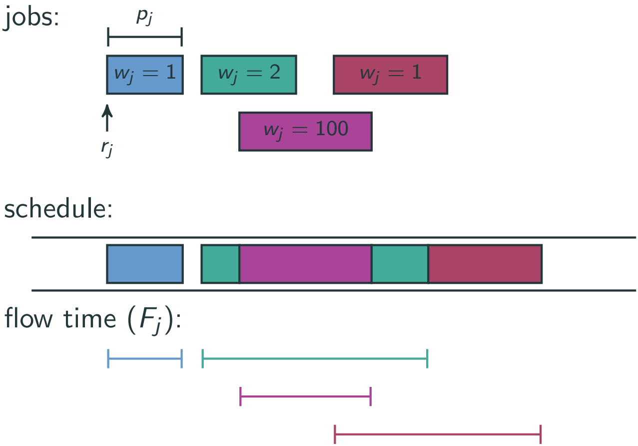

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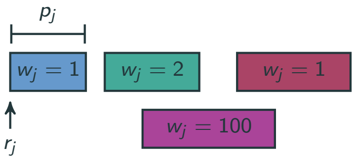
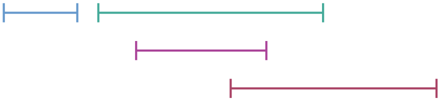


schedule:

flow time (F_j):



objective:

minimize $\sum_j w_j F_j$

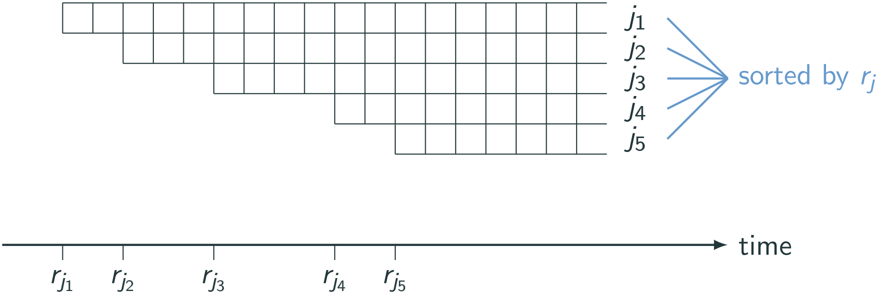
- Batra, Garg, Kumar (STOC'18): Framework and first $O(1)$-approximation (pseudo-poly. time)
- Feige, Kulkarni, Li (SODA'19): Assume w.l.o.g. that input numbers are polynomially bounded
- R., Wiese (STOC'21): $(2+\epsilon)$-approximation
- Armbruster, R., Wiese (unpublished): PTAS

Covering integer program

$$x_{j,t} \equiv \text{job } j \text{ has not finished at time } t$$

$$\begin{aligned}
&\min \sum_{j\in J}\sum_{t\geq r_j} w_j x_{j,t} \\
&\sum_{\substack{j\in J\\ s\leq r_j\leq t}} x_{j,t}\cdot p_j \geq \sum_{\substack{j\in J\\ s\leq r_j\leq t}} p_j - (t-s) && \forall s\leq t \\
&x_{j,t} \geq x_{j,t+1} && \forall j\in J, t>r_j \\
&x_{j,t} \in \{0,1\} && \forall j\in J, t\in\{r_j, r_j+1,\dots\}
\end{aligned}$$

From a different perspective



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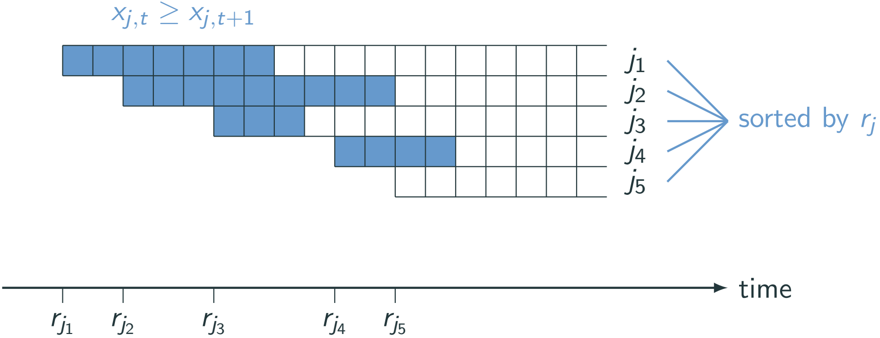

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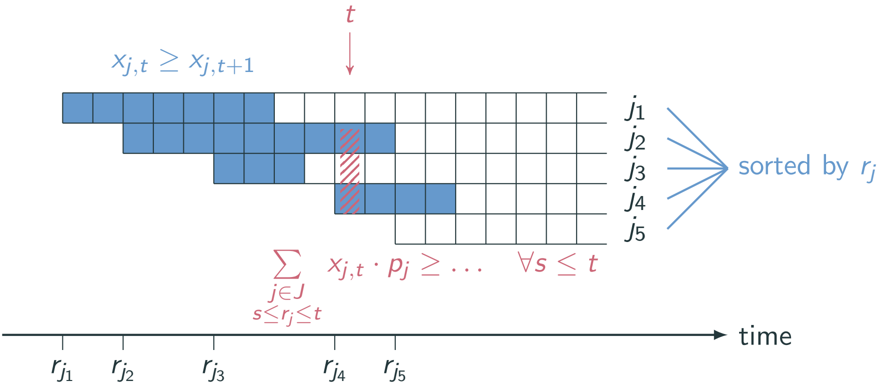
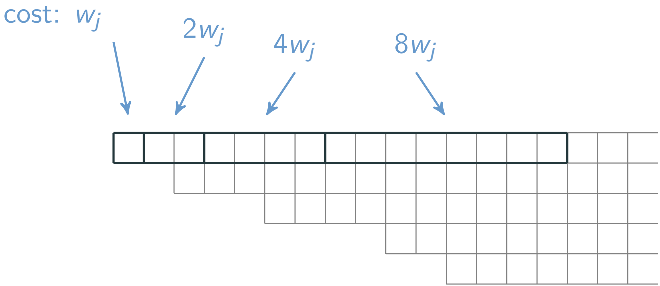


Avoiding the prefix constraint



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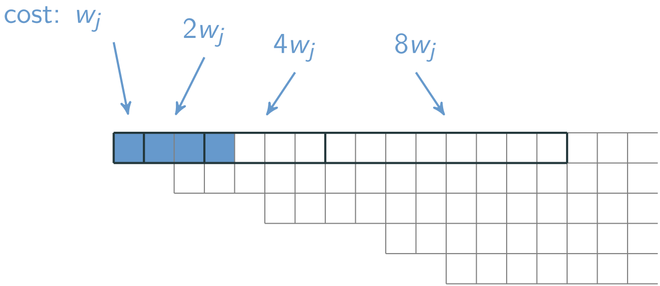

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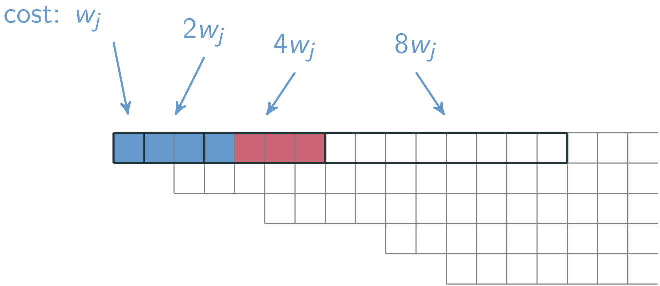

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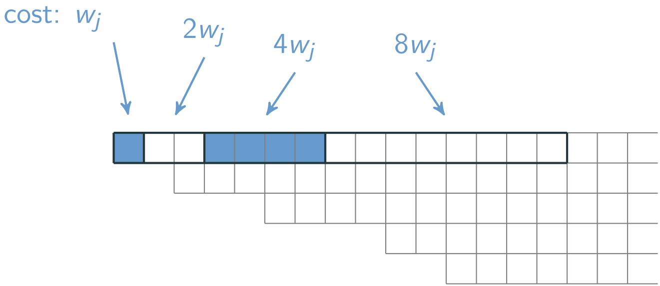

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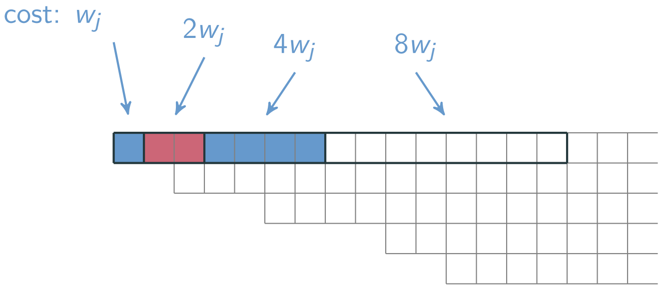

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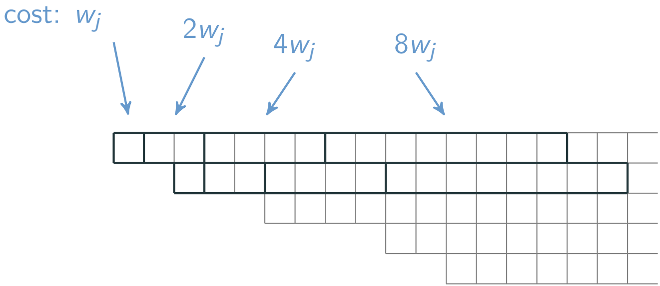

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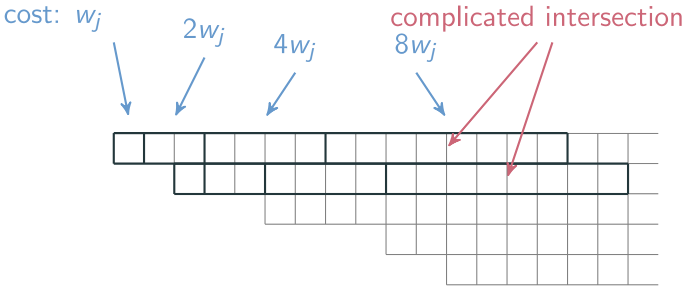
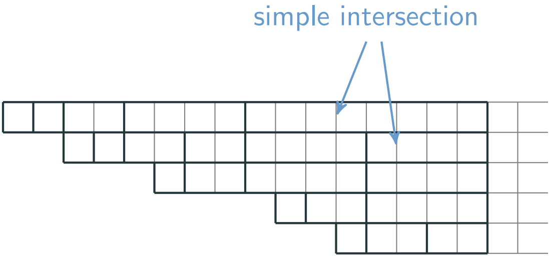


Avoiding the prefix constraint (cont'd)

Using more involved (still exponentially growing) grouping.



$\rightsquigarrow$ loses factor 32

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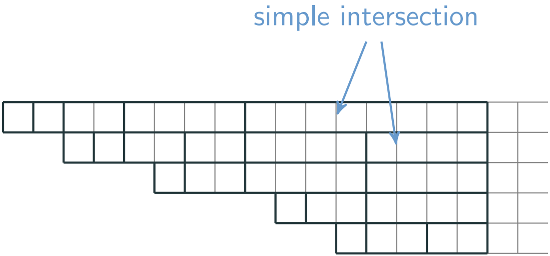
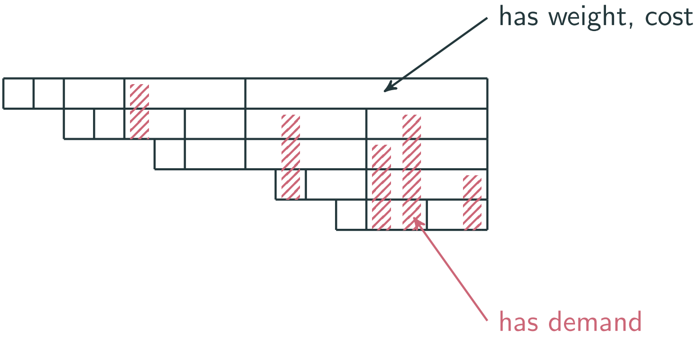


$\rightsquigarrow$ loses factor 32

can be reduced to $(1 + \epsilon)$ with some extra technicalities.

Recap



- Given hierarchically aligned rectangles
- Select a subset of rectangles of minimal cost
- Such that all demands are covered

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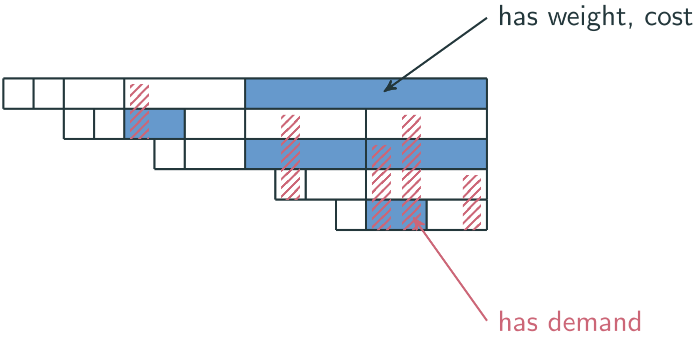

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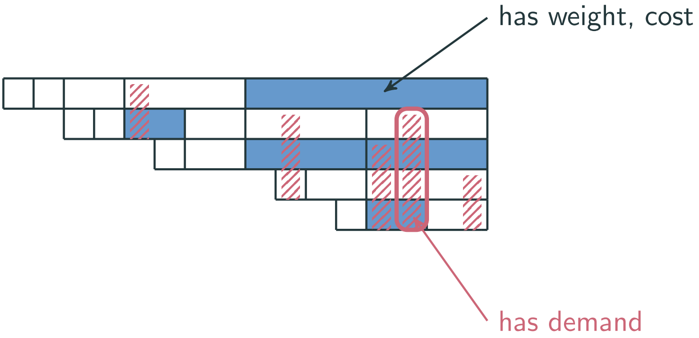

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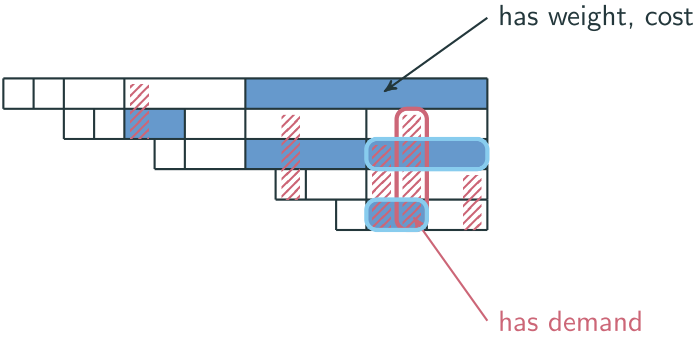

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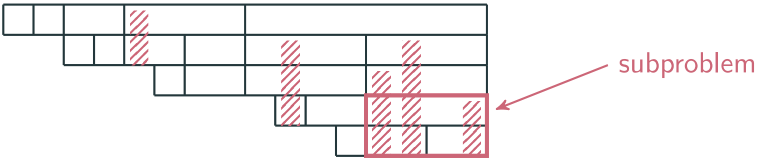
Dynamic program

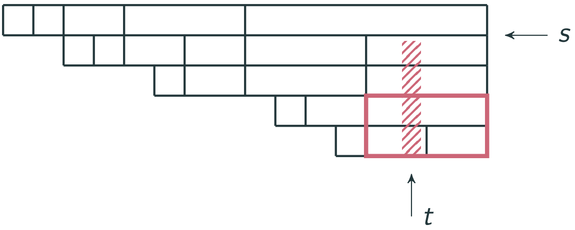
The hierarchical alignment allows for dynamic programming.



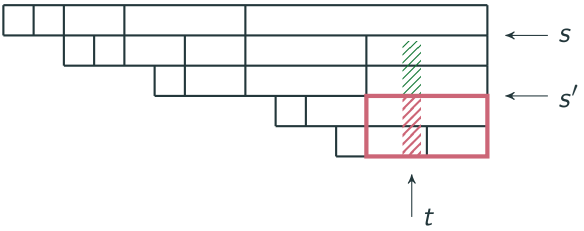
What do we need to know from the rest of the solution to determine the optimal solution of the subproblem?

Naive: how much is already covered for all demand rays that intersect the subproblem. $\rightsquigarrow$ too much information.

A closer look at the demands

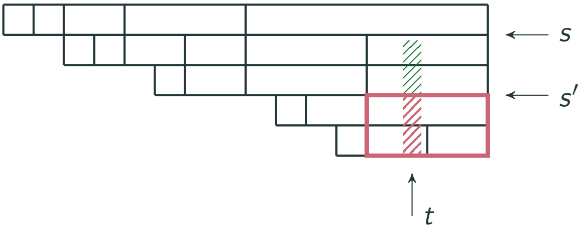


$$\sum_{\substack{j\in J: s\le r_j\le t\\ \text{rect. selected}}} p_j \ge \sum_{j\in J: s\le r_j \le t} p_j - (t-s)$$



$$\underset{\text{rect. selected}}{\sum_{j\in J:s\le r_j<s'} p_j} + \underset{\text{rect. selected}}{\sum_{j\in J:s'\le r_j\le t} p_j} \ge \sum_{j\in J:s\le r_j<s'} p_j - (s'-s) + \sum_{j\in J:s'\le r_j\le t} p_j - (t-s')$$

A closer look at the demands



$$\underbrace{\sum_{j \in J : s \le r_j < s'} p_j}_{\text{rect. selected}} + \underbrace{\sum_{j \in J : s' \le r_j \le t} p_j}_{\text{rect. selected}} \ge \sum_{j \in J : s \le r_j < s'} p_j - (s' - s) + \sum_{j \in J : s' \le r_j \le t} p_j - (t - s')$$

Fixing the solution outside the subproblem, we only need to satisfy constraint for s', t with a certain extra slack (same for all t)

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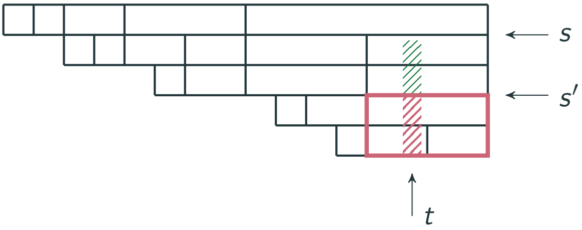

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Fixing the solution outside the subproblem, we only need to satisfy constraint for s', t with a certain extra slack (same for all t)

$\rightsquigarrow$ in DP "guess" the extra slack.

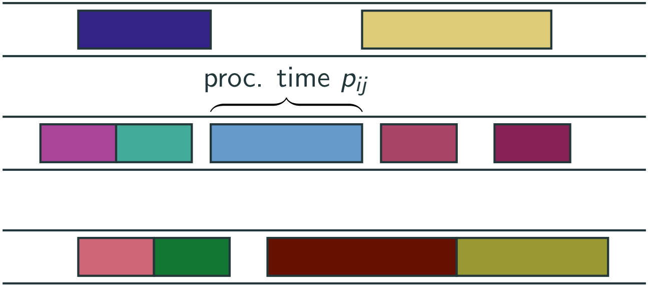
Summary

- Problem can be solved in pseudopolynomial time using DP over a tree + structural insights.
- Reduction not lossless, but error can be made $(1+\epsilon)$ using additional technical work

$\rightsquigarrow$ PTAS for sum of weighted flow time on a single machine.

parallel machines

$R \mid r_j \mid F_{\max}$



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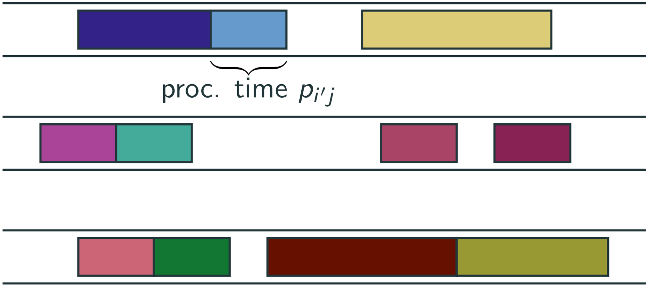
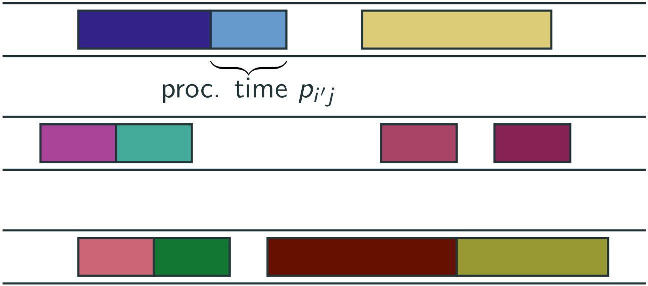


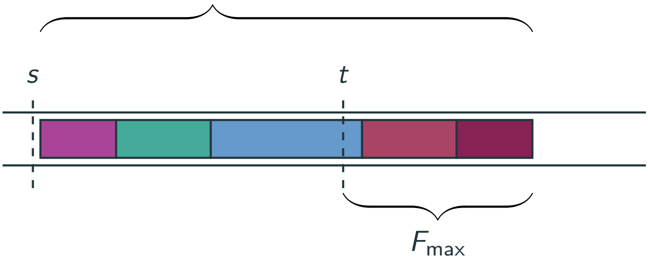


Goal: minimize max flow time $\max_j F_j$.

Condition for maximum flow time

Need to bound load on each interval of release times:

$$\sum_{\substack{s \le r_j \le t \\ j \to i}} p_{ij} \le t - s + F_{\max}.$$



Bansal-Kulkarni (STOC'15): Iterative rounding.

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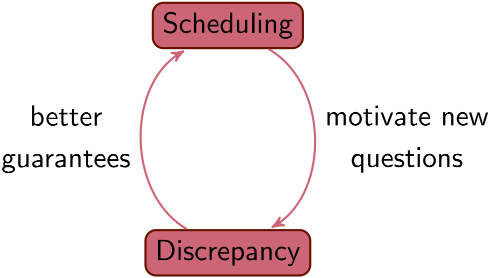
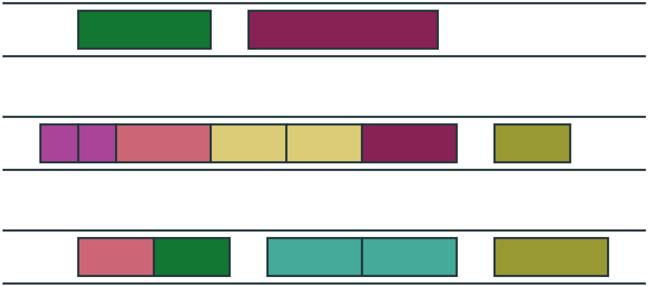
Bansal, R., Svensson (STOC'22): $O(\sqrt{\log n})$ using results from discrepancy (that rely on different methods from convex geometry).



Assume for simplicity we already have a half integral solution



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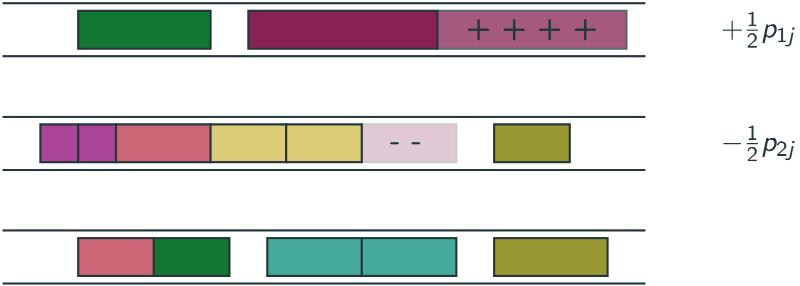

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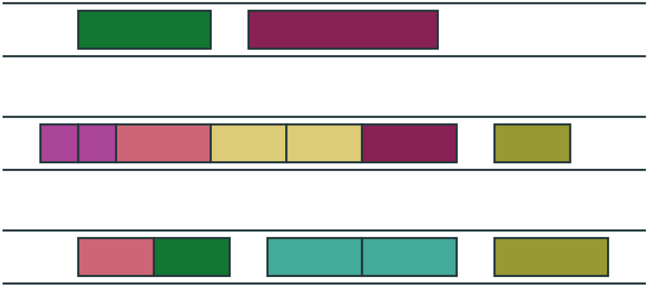

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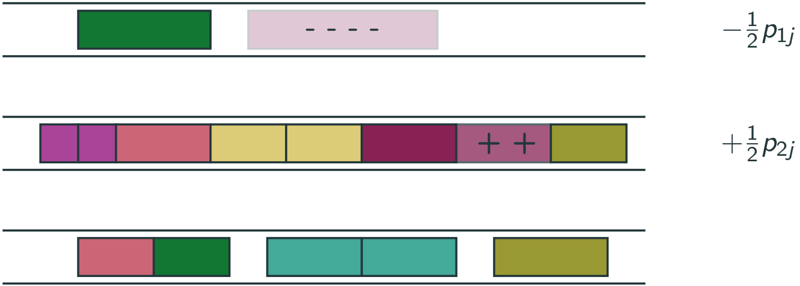

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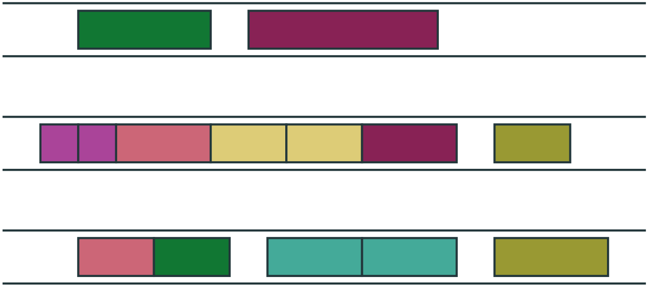

$$\pm \begin{pmatrix} \frac{1}{2} p_{1j} \\ -\frac{1}{2} p_{2j} \\ 0 \end{pmatrix}$$

Prefix Beck-Fiala

Given a series of n vectors with ℓ_1-norm $\leq T$

$$v_1 = \begin{pmatrix} 0 \\ 0.1 \\ 0 \\ -0.5 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.9 \\ -0.9 \end{pmatrix}, v_3 = \begin{pmatrix} 0.5 \\ 0 \\ -0.7 \\ 0 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \\ 0.9 \\ -0.7 \\ 0 \end{pmatrix}, \ldots$$

show there exist signs $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \ldots \in \{-1, 1\}$ s.t. for all $\ell \leq h$

$$\|\varepsilon_\ell v_\ell + \varepsilon_{\ell+1} v_{\ell+1} + \cdots + \varepsilon_h v_h\|_\infty \leq \text{“some upper bound”}$$

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Banaszczyk'98, Banaszczyk'12: $O(\sqrt{\log n} \cdot T)$ suffices

$\rightsquigarrow$ $O(\sqrt{\log n})$ integrality gap 🙂

(works also beyond the half-integral case)

Summary

Theorem
Integrality gap of LP for max flow time $\leq O(\sqrt{\log n})$*

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Integrality gap of LP for total flow time $\leq O(\log^{3/2} n)$*

(Known lower bound is $\Omega(\log n)$)

*** bounds are non-constructive, because Banaszczyk's proof does not yield an efficient algorithm**

Conclusion

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- 2.99-approximation for $P \mid r_j \mid F_{\max}$

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Thanks!