Minimizing the Weighted Number of Tardy Jobs is W[1]-hard

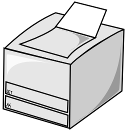
Klaus Heeger Danny Hermelin

Ben-Gurion University of the Negev

Scheduling Seminar
24.01.2024

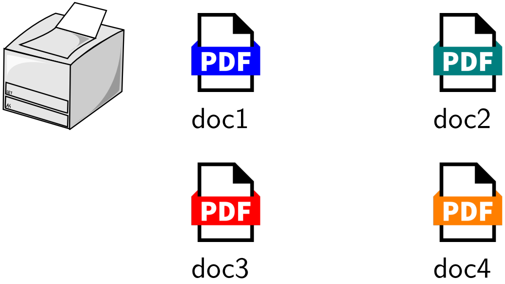
$1||\sum w_jU_j$

Machine



$1||\sum w_jU_j$

Machine **Jobs**

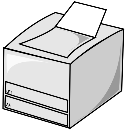


doc1 doc2

doc3 doc4

$1||\sum w_jU_j$

Machine



Jobs

doc1 $p = 5$

doc2 $p = 4$

doc3 $p = 3$

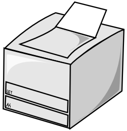
doc4 $p = 7$

Job Characteristics

p — processing time

$1||\sum w_jU_j$

Machine



Jobs

doc1: $p = 5$, $w = 4$

doc2: $p = 4$, $w = 5$

doc3: $p = 3$, $w = 6$

doc4: $p = 7$, $w = 5$

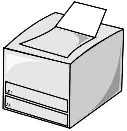
Job Characteristics

p — processing time

w — weight

$1||\sum w_jU_j$

Machine



Jobs

PDF doc1
$p = 5$
$w = 4$
$d = 16$

PDF doc2
$p = 4$
$w = 5$
$d = 6$

PDF doc3
$p = 3$
$w = 6$
$d = 3$

PDF doc4
$p = 7$
$w = 5$
$d = 19$

Job Characteristics

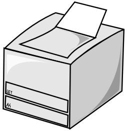
p — processing time

w — weight

d — due date

$1||\sum w_jU_j$

Machine



Jobs

PDF doc1 $p = 5$ $w = 4$ $d = 16$

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PDF doc3 $p = 3$ $w = 6$ $d = 3$

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Job Characteristics

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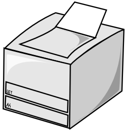
Task: Order of jobs minimizing weighted number of tardy jobs

$1||\sum w_j U_j$

Machine



Jobs

PDF doc1: $p = 5$, $w = 4$, $d = 16$

PDF doc2: $p = 4$, $w = 5$, $d = 6$

PDF doc3: $p = 3$, $w = 6$, $d = 3$

PDF doc4: $p = 7$, $w = 5$, $d = 19$

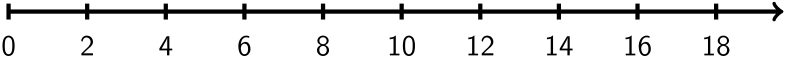
Job Characteristics

p — processing time

w — weight

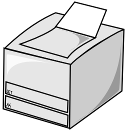
d — due date

Task: Order of jobs minimizing weighted number of tardy jobs



$1||\sum w_jU_j$

Machine



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Job Characteristics

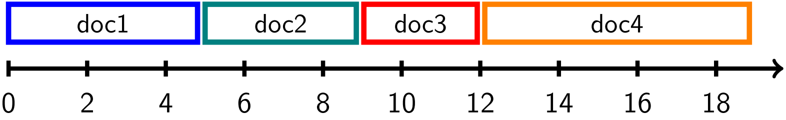
p — processing time

w — weight

d — due date

Task: Order of jobs minimizing weighted number of tardy jobs

$$0 + 5 + 6 + 0 = 11$$



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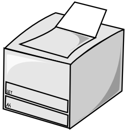

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Job Characteristics

p — processing time

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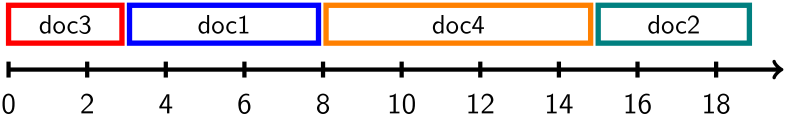
Task: Order of jobs minimizing weighted number of tardy jobs

$$0 \quad + \quad 0 \quad + \quad 0 \quad + \quad 5 \quad = 5$$



Related Work

- weakly NP-hard even if all jobs have the same due date (Knapsack) [Karp '72]

n: number of jobs

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- $\widetilde{O}(n^{p_\# + 1})$, $\widetilde{O}(n^{w_\# + 1})$ [Hermelin, Karhi, Pinedo, Shabtay '21]

$d_\#$: number of due dates

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- $2^{\widetilde{O}(p_\# + w_\#)} \cdot \mathrm{poly}(n)$, $2^{\widetilde{O}(p_\# + d_\#)} \cdot \mathrm{poly}(n)$, $2^{\widetilde{O}(d_\# + w_\#)} \cdot \mathrm{poly}(n)$ [Hermelin, Karhi, Pinedo, Shabtay '21]

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Can we improve on $\widetilde{O}(n^{w_\# + 1})$ or $\widetilde{O}(n^{p_\# + 1})$?

$d_\#$: number of due dates $p_\#$: number of processing times

$w_\#$: number of weights n: number of jobs

Main Result

Known: $\widetilde{O}(n^{p_\# + 1})$, $\widetilde{O}(n^{w_\# + 1})$-time algorithm [Hermelin, Karhi, Pinedo, Shabtay '21]

Theorem

Assuming ETH, there is no $n^{o(w_\# / \log w_\#)}$- or $n^{o(p_\# / \log p_\#)}$-time algorithm for $1||\sum w_j U_j$.

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ETH, formal: $\exists c > 1$ such that solving 3-$\textsc{Satisfiability}$ on n variables requires $O(c^n)$ time

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ETH, formal: $\exists c > 1$ such that solving 3-Satisfiability on n variables requires $O(c^n)$ time

W[1]-hardness: Under assumption FPT$\neq$W[1], no $f(w_\#) \cdot \mathrm{poly}(n)$-time algorithm/no $f(p_\#) \cdot \mathrm{poly}(n)$ -time algorithm

Reductions

A **many-one reduction** from a problem P to a problem Q is a mapping f from instances from P to instances from Q such that

1. $f(I)$ is a yes-instance if and only if I is, and
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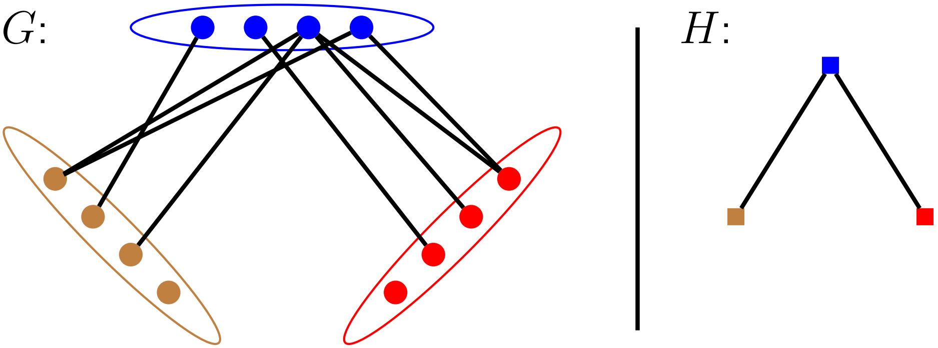
$\rightsquigarrow$ Sufficient: many-one reduction such that $w_\# = O(k)$

Multicolored Subgraph Isomorphism

Input: Two colored graphs G and H



Multicolored Subgraph Isomorphism

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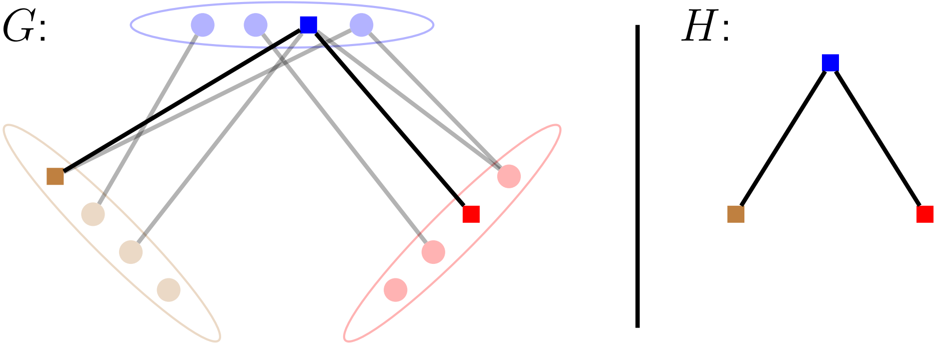

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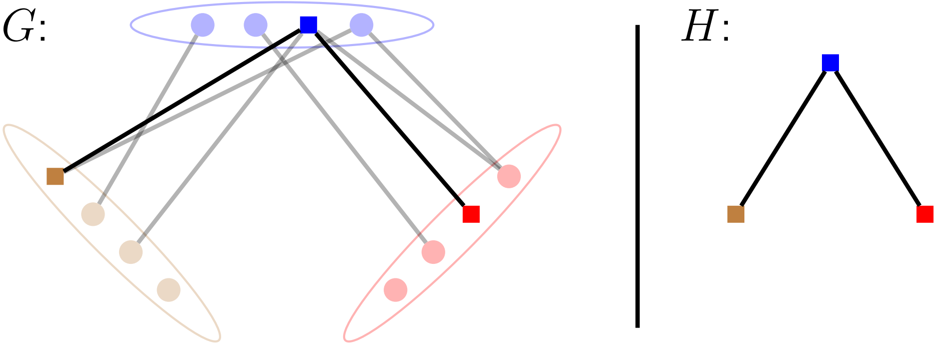
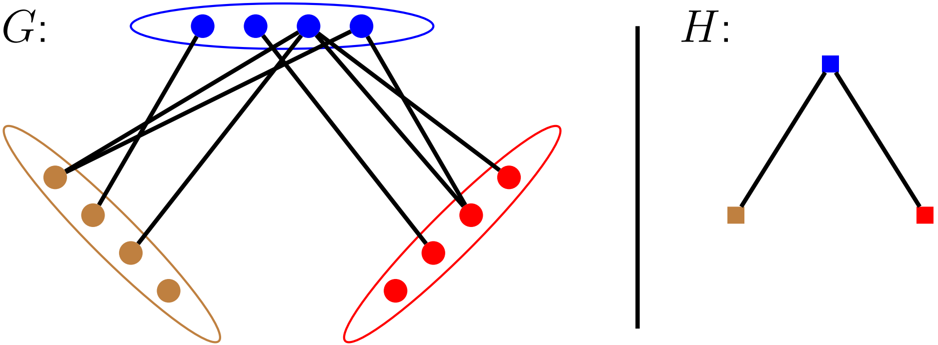


Theorem (Marx '10)

Assuming ETH, there is no $n^{o(k/\log k)}$-time algorithm for Multicolored Subgraph Isomorphism *where $k := |V(H)| + |E(H)|$.*

High-Level Idea

0. Number vertices from each color class arbitrarily



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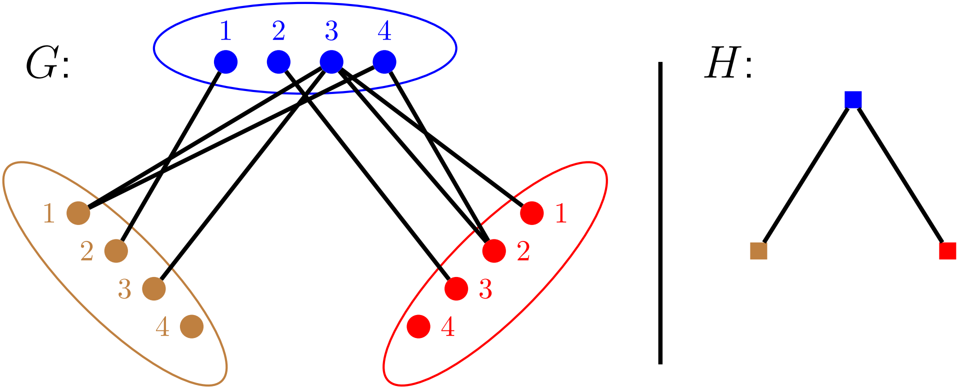

High-Level Idea

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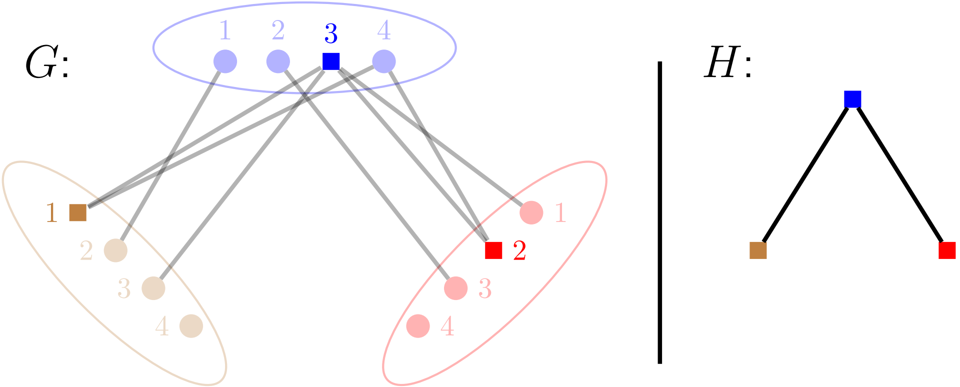

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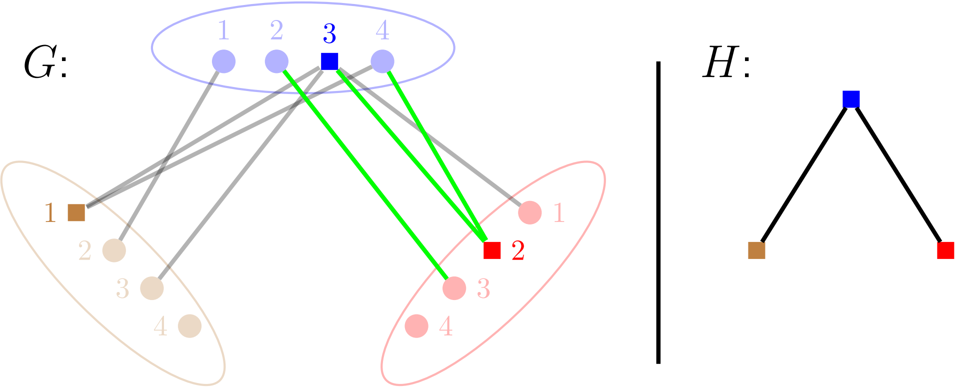
2. For each edge $\{\text{red}, \text{blue}\}$ of H, count edges $\{i', j'\}$ with $(i', j') \geq (i, i)$ (i.e. $i' > i$ or $i' = i \wedge j' \geq i$)



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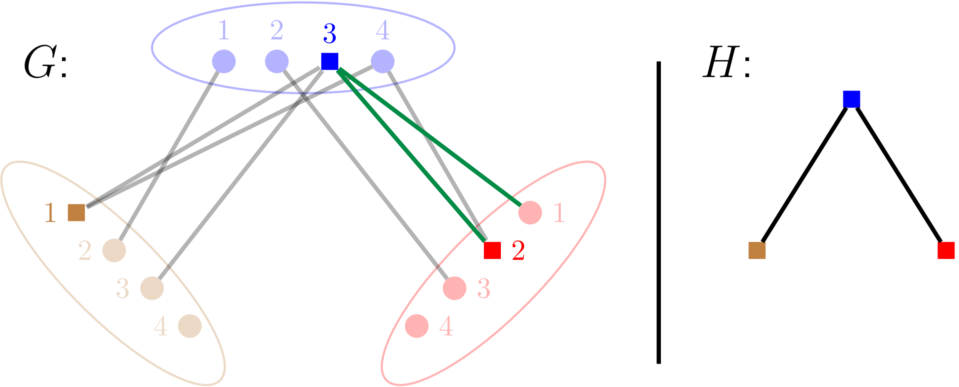
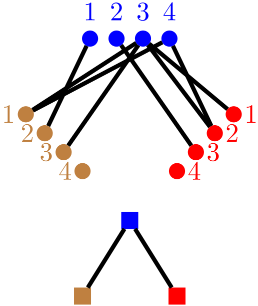


Interlude: Numbers, Digits, and Blocks

Consider numbers wrt. to some large basis N



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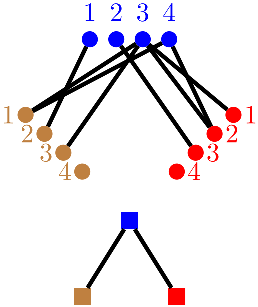
$\rightsquigarrow$ no carry-over, i.e., can treat each digit separately



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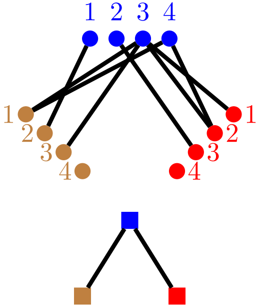
Divided into $1 + |E(H)| + |E(H)| + 1 = O(k)$ blocks:

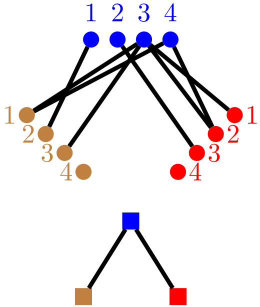
$$\underbrace{000}_{\text{vertex selection block}}\ \Big|\underbrace{\overbrace{000000}^{\text{red,blue}}\Big|\ \overbrace{000000}^{\text{brown,blue}}\ \Big|}_{\text{large blocks}}\Big|\underbrace{\overbrace{000000}^{\text{red,blue}}\Big|\ \overbrace{000000}^{\text{brown,blue}}\ \Big|}_{\text{small blocks}}0$$



Selecting Vertices

$$\boxed{w_{\#} = 2 \cdot |V(H)|}$$

For each color, two kinds of jobs J and $\neg J$ (each n times):



$$\underbrace{000}_{\text{vertex}}\ \underbrace{|000000|000000|}_{\text{large blocks}}\ \underbrace{|000000|000000|}_{\text{small blocks}}\ 0$$

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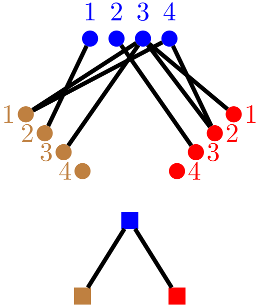
For each color, two kinds of jobs J and $\neg J$ (each n times):
Selecting $i \triangleq i \times J$ early and $(n-i) \times \neg J$ early



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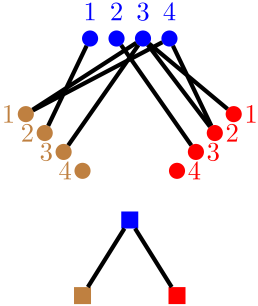
J with proc. time & weight: $001|000001|000001||000000|000000|0$



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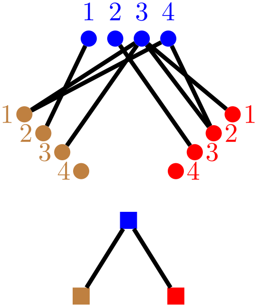

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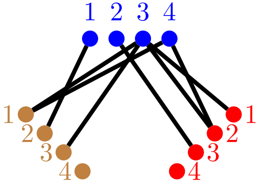
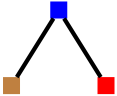
J with proc. time & weight: $001|000001|000001||000000|000000|0$

$\neg J$ with proc. time & weight: $001|000000|000000||000001|000001|0$

For both jobs, due date: $nnn|100000|000000||000000|000000|0$





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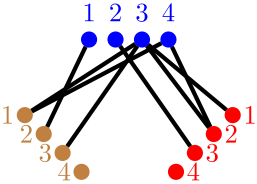
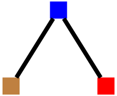
$\neg J$ with proc. time & weight: $001|000000|000000||000001|000001|0$

For both jobs, due date: $nnn|100000|000000||000000|000000|0$



After selecting $1, 2, 3$, proc. time and weight is

$$nnn\ |000023|000013||0000n-2n-3|0000n-1n-3|\ 0$$



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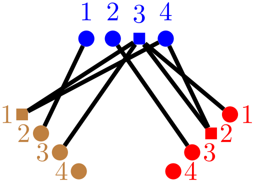
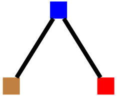
$\neg J$ with proc. time & weight: $001|000000|000000||000001|000001|0$

For both jobs, due date: $nnn|100000|000000||000000|000000|0$



After selecting $1, 2, 3$, proc. time and weight is

$$nnn\ |000023|000013||0000n-2n-3|0000n-1n-3|\ 0$$

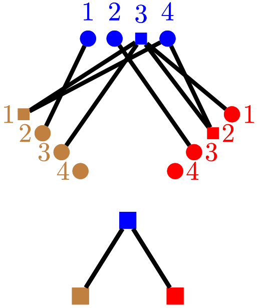


$$\underbrace{000}_{\text{vertex}}\ \underbrace{|000000|000000|}_{\text{large blocks}}\ \underbrace{|000000|000000|}_{\text{small blocks}}\ 0$$

Large Blocks—(red, blue)-block

Recall: Want to “count” edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$

$$\boxed{w_{\#} = 2 \cdot |V(H)|}$$



weight: $nnn|000023|000013||0000n-2n-3|0000n-1n-3|0$

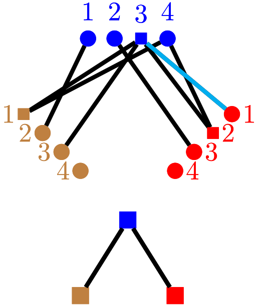
processing time: $\underbrace{nnn}_{\text{vertex}}\underbrace{|000023|000013|}_{\text{large blocks}}\underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}0$

Large Blocks—(red, blue)-block

$$\boxed{w_{\#} = 2 \cdot |V(H)|}$$

Recall: Want to "count" edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$

For first edge $e = \{1, 3\}$, two jobs:



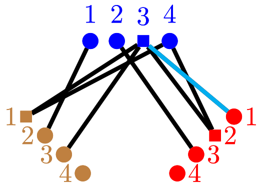
weight: $nnn|000023|000013||0000n-2n-3|0000n-1n-3|0$

processing time: $\underbrace{nnn}_{\text{vertex}}\underbrace{|000023|000013|}_{\text{large blocks}}\underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}0$

Large Blocks—(red, blue)-block

$$\boxed{w_{\#} = 2 \cdot |V(H)|}$$

Recall: Want to “count” edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$

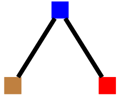


For first edge $e = \{1, 3\}$, two jobs:

J^e with weight $|100000|000000||000000|000000|1$,

processing time $|000100|000000||000000|000000|0$,

due date $nnn|000113|100000||000000|000000|0$.



weight: $nnn|000023|000013||0000n-2n-3|0000n-1n-3|0$

processing time: $\underbrace{nnn}_{\text{vertex}}\underbrace{|000023|000013|}_{\text{large blocks}}\underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}0$

Large Blocks—(red, blue)-block

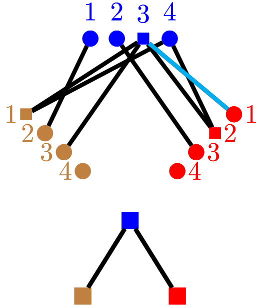
$$w_\# = 2 \cdot |V(H)| + 2$$

Recall: Want to “count” edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$

For first edge $e = \{1, 3\}$, two jobs:

J^e with weight $|100000|000000||000000|000000|1$,
processing time $|000100|000000||000000|000000|0$,
due date $nnn|000113|100000||000000|000000|0$.

$\neg J^e$ w. weight $|100000|000000||000000|000000|0$,
processing time $|000100|000000||000000|000000|0$,
due date $nnn|0001nn|100000||000000|000000|0$

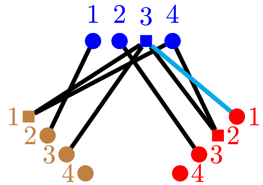


weight: $nnn|000023|000013||0000n-2n-3|0000n-1n-3|0$

processing time: $\underbrace{nnn}_{\text{vertex}}\underbrace{|000023|000013|}_{\text{large blocks}}\underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}0$

Large Blocks—(red, blue)-block

$$w_{\#} = 2 \cdot |V(H)| + 2$$



Recall: Want to "count" edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$

For first edge $e = \{1, 3\}$, two jobs:

J^e with weight $|100000|000000||000000|000000|1$,

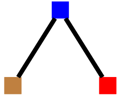
processing time $|000100|000000||000000|000000|0$,

due date $nnn|000113|100000||000000|000000|0$.

$\neg J^e$ w. weight $|100000|000000||000000|000000|0$,

processing time $|000100|000000||000000|000000|0$,

due date $nnn|0001nn|100000||000000|000000|0$



$\Rightarrow$ exactly one of J^e and $\neg J^e$ can be early

weight: $nnn|000023|000013||0000n-2n-3|0000n-1n-3|0$

processing time: $\underbrace{nnn}_{\text{vertex}}\underbrace{|000023|000013|}_{\text{large blocks}}\underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}0$

Large Blocks—(red, blue)-block

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 2}$$

Recall: Want to "count" edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$

For first edge $e = \{1, 3\}$, two jobs:

J^e with weight $|100000|000000||000000|000000|1,$

processing time $|000100|000000||000000|000000|0,$

due date $nnn|000113|100000||000000|000000|0.$

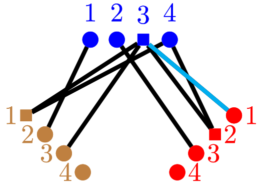
$\neg J^e$ w. weight $|100000|000000||000000|000000|0,$

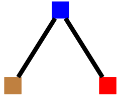
processing time $|000100|000000||000000|000000|0,$

due date $nnn|0001nn|100000||000000|000000|0$

$\Rightarrow$ exactly one of J^e and $\neg J^e$ can be early

J^e can be early if $(1, 3) \geq (2, 3)$





weight: $nnn|000023|000013||0000n-2n-3|0000n-1n-3|0$

$$\text{processing time: } \underbrace{nnn}_{\text{vertex}}\ \underbrace{|000023|000013|}_{\text{large blocks}}\ \underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}\ 0$$

Large Blocks—(red, blue)-block

$w_{\#} = 2 \cdot |V(H)| + 2$

Recall: Want to "count" edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$

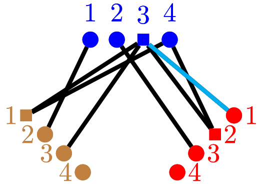
For first edge $e = \{1, 3\}$, two jobs:

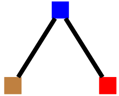
J^e with weight $|100000|000000||000000|000000|1$,
processing time $|000100|000000||000000|000000|0$,
due date $nnn|000113|100000||000000|000000|0$.

$\neg J^e$ w. weight $|100000|000000||000000|000000|0$,
processing time $|000100|000000||000000|000000|0$,
due date $nnn|0001nn|100000||000000|000000|0$

$\Rightarrow$ exactly one of J^e and $\neg J^e$ can be early

J^e can be early if $(1, 3) \geq (2, 3)$





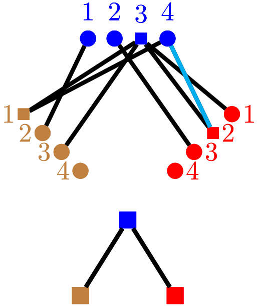
weight: $nnn|100023|000013||0000n-2n-3|0000n-1n-3|0$

processing time: $\underbrace{nnn}_{\text{vertex}}\underbrace{|000123|000013|}_{\text{large blocks}}\underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}0$

Large Blocks—(red, blue)-block, 2nd edge

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 2}$$

Recall: Want to "count" edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$



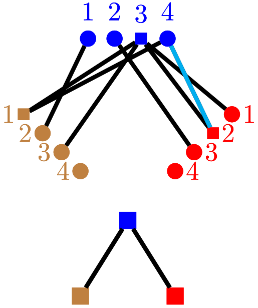
$$\text{weight: } nnn|100023|000013||0000n-2n-3|0000n-1n-3|0$$

$$\text{processing time } \underbrace{nnn}_{\text{vertex}}\ \underbrace{|000123|000013|}_{\text{large blocks}}\ \underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}\ 0$$

Large Blocks—(red, blue)-block, 2nd edge

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 2}$$

Recall: Want to “count” edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$



For second edge $e = \{2, 4\}$, two jobs:

J^e with weight $|100000|000000||000000|000000|1$,
processing time $|001000|000000||000000|000000|0$, and
due date $nnn|001124|100000||000000|000000|0$.

$\neg J^e$ w. weight $|100000|000000||000000|000000|0$,
processing time $|001000|000000||000000|000000|0$, and
due date $nnn|0011nn|100000||000000|000000|0$.

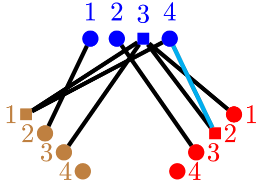
weight: $nnn|100023|000013||0000n-2n-3|0000n-1n-3|0$

processing time $\underbrace{nnn}_{\text{vertex}}\ \underbrace{|000123|000013|}_{\text{large blocks}}\ \underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}\ 0$

Large Blocks—(red, blue)-block, 2nd edge

$$w_{\#} = 2 \cdot |V(H)| + 2$$

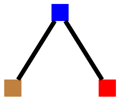
Recall: Want to “count” edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$



For second edge $e = \{2, 4\}$, two jobs:

J^e with weight $|100000|000000||000000|000000|1$,
processing time $|001000|000000||000000|000000|0$, and
due date $nnn|001124|100000||000000|000000|0$.

$\neg J^e$ w. weight $|100000|000000||000000|000000|0$,
processing time $|001000|000000||000000|000000|0$, and
due date $nnn|0011nn|100000||000000|000000|0$.



$\Rightarrow$ one of J^e and $\neg J^e$ can be scheduled early.

J^e can be early if $(2, 4) \geq (2, 3)$

weight: $nnn|100023|000013||0000n-2n-3|0000n-1n-3|0$

$$\text{processing time } \underbrace{nnn}_{\text{vertex}}\ \underbrace{|000123|000013|}_{\text{large blocks}}\ \underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}\ 0$$

Large Blocks—(red, blue)-block, 2nd edge

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 2}$$

Recall: Want to "count" edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$

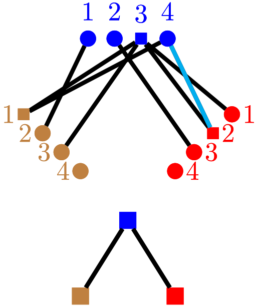
For second edge $e = \{2, 4\}$, two jobs:

J^e with weight $|100000|000000||000000|000000|1$,
processing time $|001000|000000||000000|000000|0$, and
due date $nnn|001124|100000||000000|000000|0$.

$\neg J^e$ w. weight $|100000|000000||000000|000000|0$,
processing time $|001000|000000||000000|000000|0$, and
due date $nnn|0011nn|100000||000000|000000|0$.

$\Rightarrow$ one of J^e and $\neg J^e$ can be scheduled early.

J^e can be early if $(2, 4) \geq (2, 3)$



weight: $nnn|200023|000013||0000n-2n-3|0000n-1n-3|1$

processing time $\underbrace{nnn}_{\text{vertex}}\ \underbrace{|001123|000013|}_{\text{large blocks}}\ \underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}\ 0$

Large Blocks—(red, blue)-block, 3rd edge

$$\boxed{w_\# = 2 \cdot |V(H)| + 2}$$

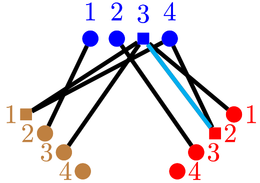
Recall: Want to “count” edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$

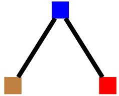
For third edge $e = \{2, 3\}$, two jobs:

J^e with weight $|100000|000000||000000|000000|1$,
processing time $|010000|000000||000000|000000|0$, and
due date $nnn|011123|100000||000000|000000|0$.

$\neg J^e$ w. weight $|100000|000000||000000|000000|0$,
processing time $|010000|000000||000000|000000|0$, and
due date $nnn|0111nn|100000||000000|000000|0$.

$\Rightarrow$ one of J^e and $\neg J^e$ can be scheduled early.
J^e can be early if $(2, 3) \geq (2, 3)$





weight: $nnn|200023|000013||0000n-2n-3|0000n-1n-3|1$

processing time $\underbrace{nnn}_{\text{vertex}}\ \underbrace{|001123|000013|}_{\text{large blocks}}\ \underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}\ 0$

Large Blocks—(red, blue)-block, 4th edge

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 2}$$

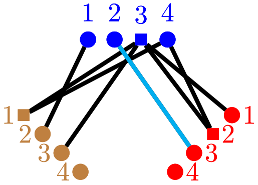
Recall: Want to "count" edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$

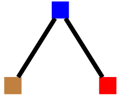
For fourth edge $e = \{3, 2\}$, two jobs:

J^e with weight $|100000|000000||000000|000000|1$,
processing time $|100000|000000||000000|000000|0$, and
due date $nnn|111132|100000||000000|000000|0$.

$\neg J^e$ w. weight $|100000|000000||000000|000000|0$,
processing time $|100000|000000||000000|000000|0$, and
due date $nnn|1111nn|100000||000000|000000|0$.

$\Rightarrow$ one of J^e and $\neg J^e$ can be scheduled early.
J^e can be early if $(3, 2) \geq (2, 3)$





weight: $nnn|300023|000013||0000n-2n-3|0000n-1n-3|2$

$$\text{processing time } \underbrace{nnn}_{\text{vertex}}\ \underbrace{|011123|000013|}_{\text{large blocks}}\ \underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}\ 0$$

Large Blocks—(red, blue)-block, 4th edge

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 2}$$

Recall: Want to "count" edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$

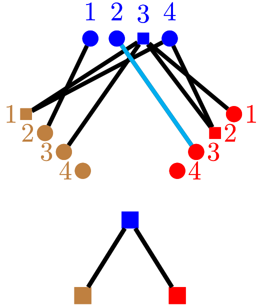
For fourth edge $e = \{3, 2\}$, two jobs:

J^e with weight $|100000|000000||000000|000000|1$,
processing time $|100000|000000||000000|000000|0$, and
due date $nnn|111132|100000||000000|000000|0$.

$\neg J^e$ w. weight $|100000|000000||000000|000000|0$,
processing time $|100000|000000||000000|000000|0$, and
due date $nnn|1111nn|100000||000000|000000|0$.

$\Rightarrow$ one of J^e and $\neg J^e$ can be scheduled early.

J^e can be early if $(3, 2) \geq (2, 3)$



weight: $nnn|400023|000013||0000n-2n-3|0000n-1n-3|3$

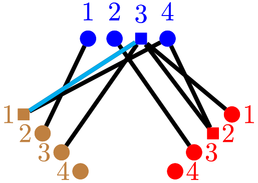
processing time $\underbrace{nnn}_{\text{vertex}}\ \underbrace{|111123|000013|}_{\text{large blocks}}\ \underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}\ 0$

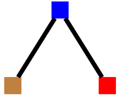
Large Blocks—(brown, blue)-block

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 2}$$

Recall: Want to “count” edges $\{i', k'\}$ with $(i', k') \geq (1, 3)$

For first edge $e = \{1, 3\}$, two jobs:





weight: $nnn|m00023|000013||0000n-2n-3|0000n-1n-3|3$

processing time: $\underbrace{nnn}_{\text{vertex}}\underbrace{|111123|000013|}_{\text{large blocks}}\underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}0$

Large Blocks—(brown, blue)-block

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 2}$$

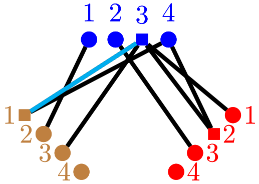
Recall: Want to "count" edges $\{i', k'\}$ with $(i', k') \geq (1, 3)$

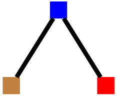
For first edge $e = \{1, 3\}$, two jobs:

J^e with weight $|100000||000000|000000|1,$

processing time $|000100||000000|000000|0,$

due date $nnn|111123|100000||000000|000000|0.$





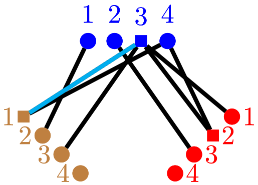
weight: $nnn|m00023|000013||0000n-2n-3|0000n-1n-3|3$

processing time: $\underbrace{nnn}_{\text{vertex}}\underbrace{|111123|000013|}_{\text{large blocks}}\underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}0$

Large Blocks—(brown, blue)-block

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 2}$$

Recall: Want to "count" edges $\{i', k'\}$ with $(i', k') \geq (1, 3)$



For first edge $e = \{1, 3\}$, two jobs:

J^e with weight $|100000||000000|000000|1,$

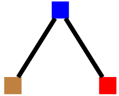
processing time $|000100||000000|000000|0,$

due date $nnn|111123|100000||000000|000000|0.$

$\neg J^e$ w. weight $|100000||000000|000000|0,$

processing time $|000100||000000|000000|0,$

due date $nnn|111123|0001nn||100000|000000|0$



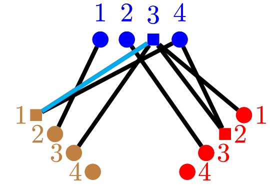
weight: $nnn|m00023|000013||0000n-2n-3|0000n-1n-3|3$

processing time: $\underbrace{nnn}_{\text{vertex}}\,\underbrace{|111123|000013|}_{\text{large blocks}}\,\underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}\,0$

Large Blocks—(brown, blue)-block

$\boxed{w_\# = 2 \cdot |V(H)|+2}$

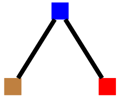
Recall: Want to “count” edges $\{i', k'\}$ with $(i', k') \geq (1, 3)$



For first edge $e = \{1, 3\}$, two jobs:

J^e with weight $|100000||000000|000000|1$,
processing time $|000100||000000|000000|0$,
due date $nnn|111123|100000||000000|000000|0$.

$\neg J^e$ w. weight $|100000||000000|000000|0$,
processing time $|000100||000000|000000|0$,
due date $nnn|111123|0001nn||100000|000000|0$

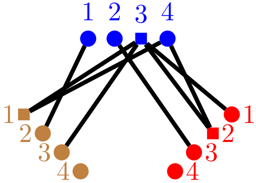


$\Rightarrow$ J^e can be early if ???

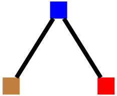
weight: $nnn|m00023|000013||0000n-2n-3|0000n-1n-3|3$
processing time: $\underbrace{nnn}_{\text{vertex}}|\underbrace{111123|000013|}_{\text{large blocks}}|\underbrace{0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}\,0$

Large Blocks—(red, blue)-block, after last edge

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 2}$$



Recall: Want to "count" edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$

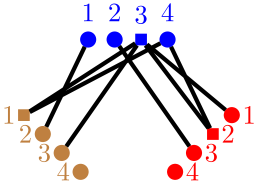


weight: $nnn|m00023|000013||0000n-2n-3|0000n-1n-3|3$

processing time: $nnn\ |111123|000013||0000n-2n-3|0000n-1n-3|\ 0$

Large Blocks—(red, blue)-block, after last edge

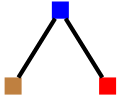
$$\boxed{w_{\#} = 2 \cdot |V(H)| + 2}$$



Recall: Want to "count" edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$

Two kinds of filler jobs, each n times:

$J_{\text{red}}^{(\text{red},\text{blue})}$ with weight & proc. time $|000010|000000||000000|000000|0$



weight: $nnn|m00023|000013||0000n-2n-3|0000n-1n-3|3$

processing time: $nnn\ |111123|000013||0000n-2n-3|0000n-1n-3|\ 0$

Large Blocks—(red, blue)-block, after last edge

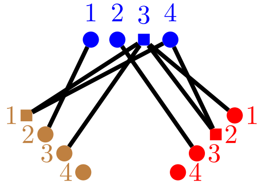
$$\boxed{w_{\#} = 2 \cdot |V(H)| + 2}$$

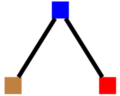
Recall: Want to "count" edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$

Two kinds of filler jobs, each n times:

$J^{(\text{red},\text{blue})}_{\text{red}}$ with weight & proc. time $|000010|000000||000000|000000|0$

$J^{(\text{red},\text{blue})}_{\text{blue}}$ with weight & proc. time $|000001|000000||000000|000000|0$





weight: $nnn|m00023|000013||0000n-2n-3|0000n-1n-3|3$

processing time: $nnn\ |111123|000013||0000n-2n-3|0000n-1n-3|\ 0$

Large Blocks—(red, blue)-block, after last edge

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 2}$$

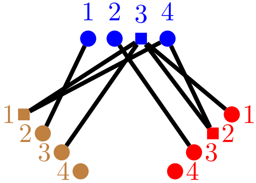
Recall: Want to "count" edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$

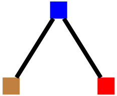
Two kinds of filler jobs, each n times:

$J^{(\text{red},\text{blue})}_{\text{red}}$ with weight & proc. time $|000010|000000||000000|000000|0$

$J^{(\text{red},\text{blue})}_{\text{blue}}$ with weight & proc. time $|000001|000000||000000|000000|0$

due date $nnn|1111nn|100000||000000|000000|0$



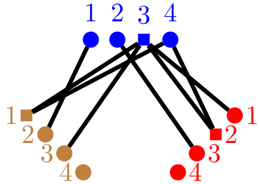


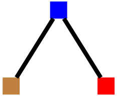
weight: $nnn|m00023|000013||0000n-2n-3|0000n-1n-3|3$

processing time: $nnn\,|111123|000013||0000n-2n-3|0000n-1n-3|\ 0$

Large Blocks—(red, blue)-block, after last edge

$$\boxed{w_\# = 2\cdot|V(H)|+4}$$





Recall: Want to “count” edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$

Two kinds of filler jobs, each n times:

$J^{(\text{red},\text{blue})}_{\text{red}}$ with weight & proc. time $|000010|000000||000000|000000|0$

$J^{(\text{red},\text{blue})}_{\text{blue}}$ with weight & proc. time $|000001|000000||000000|000000|0$

due date $nnn|1111nn|100000||000000|000000|0$

weight: $nnn|m00023|000013||0000n-2n-3|0000n-1n-3|3$

processing time: $nnn\ |111123|000013||0000n-2n-3|0000n-1n-3|\ 0$

Large Blocks—(red, blue)-block, after last edge

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 4}$$

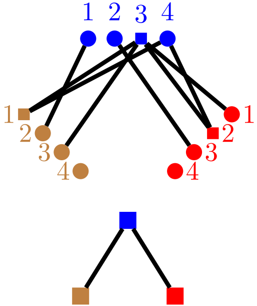
Recall: Want to “count” edges $\{j', k'\}$ with $(j', k') \geq (2, 3)$

Two kinds of filler jobs, each n times:

$J^{(\text{red},\text{blue})}_{\text{red}}$ with weight & proc. time $|000010|000000||000000|000000|0$

$J^{(\text{red},\text{blue})}_{\text{blue}}$ with weight & proc. time $|000001|000000||000000|000000|0$

due date $nnn|1111nn|100000||000000|000000|0$



weight: $nnn|m000nn|000013||0000n-2n-3|0000n-1n-3|3$

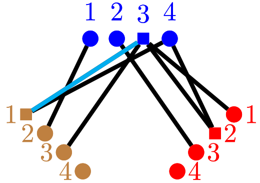
processing time: $nnn\ |1111nn|000013||0000n-2n-3|0000n-1n-3|\ 0$

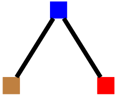
Large Blocks—(brown, blue)-block

$$w_{\#} = 2 \cdot |V(H)| + 4$$

Recall: Want to "count" edges $\{i', k'\}$ with $(i', k') \geq (1, 3)$

For first edge $e = \{1, 3\}$, two jobs:





weight: $nnn|m000nn|000013||0000n-2n-3|0000n-1n-3|3$

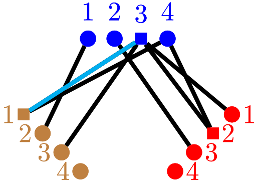
processing time: $\underbrace{nnn}_{\text{vertex}}\underbrace{|111123|000013|}_{\text{large blocks}}\underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}0$

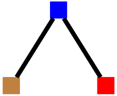
Large Blocks—(brown, blue)-block

$$w_{\#} = 2 \cdot |V(H)| + 6$$

Recall: Want to "count" edges $\{i', k'\}$ with $(i', k') \geq (1, 3)$

For first edge $e = \{1, 3\}$, two jobs:





weight: $nnn|m000nn|000013||0000n-2n-3|0000n-1n-3|3$

processing time: $\underbrace{nnn}_{\text{vertex}}\underbrace{|111123|000013|}_{\text{large blocks}}\underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}0$

Large Blocks—(brown, blue)-block

$$\boxed{w_\# = 2\cdot|V(H)|+6}$$

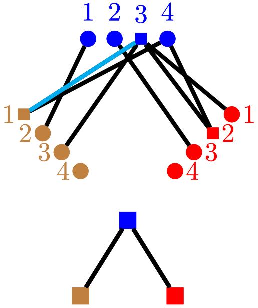
Recall: Want to “count” edges $\{i', k'\}$ with $(i', k') \geq (1, 3)$

For first edge $e = \{1, 3\}$, two jobs:

J^e with weight $|100000||000000|000000|1,$

processing time $|000100||000000|000000|0,$

due date $nnn|1111nn|000113||000000|000000|0.$



weight: $nnn|m000nn|000013||0000n-2n-3|0000n-1n-3|3$

processing time: $\underbrace{nnn}_{\text{vertex}}\underbrace{|111123|000013|}_{\text{large blocks}}\underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}0$

Large Blocks—(brown, blue)-block

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 6}$$

Recall: Want to “count” edges $\{i', k'\}$ with $(i', k') \geq (1, 3)$

For first edge $e = \{1, 3\}$, two jobs:

J^e with weight $|100000||000000|000000|1,$

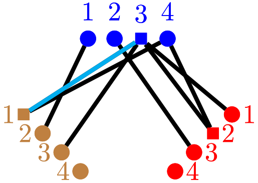
processing time $|000100||000000|000000|0,$

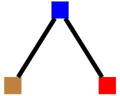
due date $nnn|1111nn|000113||000000|000000|0.$

$\neg J^e$ w. weight $|100000||000000|000000|0,$

processing time $|000100||000000|000000|0,$

due date $nnn|1111nn|0001nn||100000|000000|0$





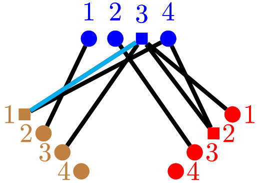
weight: $nnn|m000nn|000013||0000n-2n-3|0000n-1n-3|3$

processing time: $\underbrace{nnn}_{\text{vertex}}\ \underbrace{|111123|000013|}_{\text{large blocks}}\ \underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}\ 0$

Large Blocks—(brown, blue)-block

$$\boxed{w_\# = 2 \cdot |V(H)| + 6}$$

Recall: Want to "count" edges $\{i', k'\}$ with $(i', k') \geq (1, 3)$



For first edge $e = \{1, 3\}$, two jobs:

J^e with weight $|100000||000000|000000|1,$

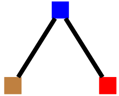
processing time $|000100||000000|000000|0,$

due date $nnn|1111nn|000113||000000|000000|0.$

$\neg J^e$ w. weight $|100000||000000|000000|0,$

processing time $|000100||000000|000000|0,$

due date $nnn|1111nn|0001nn||100000|000000|0$



$\Rightarrow$ J^e can be early if $(1, 3) \geq (1, 3)$

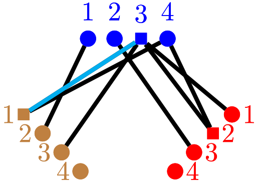
weight: $nnn|m000nn|000013||0000n-2n-3|0000n-1n-3|3$

processing time: $\underbrace{nnn}_{\text{vertex}}\,\underbrace{|111123|000013|}_{\text{large blocks}}\,\underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}\,0$

Large Blocks—(brown, blue)-block

$$\boxed{w_\# = 2 \cdot |V(H)| + 6}$$

Recall: Want to "count" edges $\{i', k'\}$ with $(i', k') \geq (1, 3)$



For first edge $e = \{1, 3\}$, two jobs:

J^e with weight $|100000||000000|000000|1,$

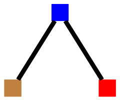
processing time $|000100||000000|000000|0,$

due date $nnn|1111nn|000113||000000|000000|0.$

$\neg J^e$ w. weight $|100000||000000|000000|0,$

processing time $|000100||000000|000000|0,$

due date $nnn|1111nn|0001nn||100000|000000|0$



$\Rightarrow$ J^e can be early if $(1, 3) \geq (1, 3)$

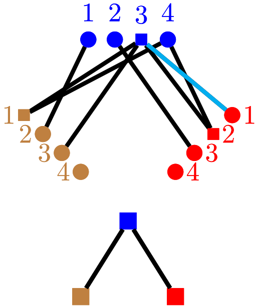
exactly one of J^e and $\neg J^e$ can be early.

$$\begin{array}{ll} \text{weight:} & nnn|m000nn|000013||0000n-2n-3|0000n-1n-3|3 \\ \text{processing time:} & \underbrace{nnn}_{\text{vertex}}\,\underbrace{|111123|000013|}_{\text{large blocks}}\,\underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}\,0 \end{array}$$

Small Blocks—(red, blue)-block

Now: Want to "count" edges $\{j, k\}$ with $(j, k) \leq (2, 3)$

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 4 \cdot |E(H)|}$$



weight: $nnn|m000nn|m000nn||0000n-2n-3|0000n-1n-3|7$

processing time: $\underbrace{nnn}_{\text{vertex}}\ \underbrace{|1111nn|1111nn|}_{\text{large blocks}}\ \underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}\ 0$

Small Blocks—(red, blue)-block

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 4 \cdot |E(H)|}$$

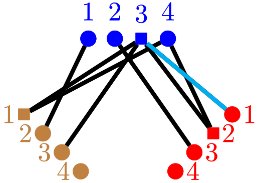
Now: Want to "count" edges $\{j, k\}$ with $(j, k) \leq (2, 3)$

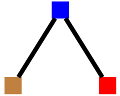
For first edge $e = \{1, 3\}$, two jobs:

J^e with weight $|1000 \quad 0 \quad 0|000000|1$

processing time $|0001 \quad 0 \quad 0|000000|0$

due date $nnn|1111nn|1111nn||0001n-2n-3|100000|0$





weight: $nnn|m000nn|m000nn||0000n-2n-3|0000n-1n-3|7$

processing time: $\underbrace{nnn}_{\text{vertex}} \ \underbrace{|1111nn|1111nn|}_{\text{large blocks}} \ \underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}} \ 0$

Small Blocks—(red, blue)-block

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 4 \cdot |E(H)|}$$

Now: Want to "count" edges $\{j, k\}$ with $(j, k) \leq (2, 3)$

For first edge $e = \{1, 3\}$, two jobs:

J^e with weight $|1000 \quad 0 \quad 0|000000|1$

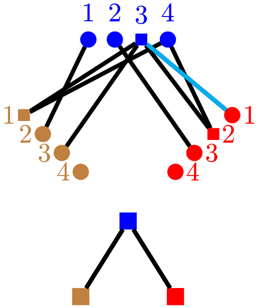
processing time $|0001 \quad 0 \quad 0|000000|0$

due date $nnn|1111nn|1111nn||0001n - 2n - 3|100000|0$

$\neg J^e$ with weight $|1000 \quad 0 \quad 0|000000|0$

processing time $|0001 \quad 0 \quad 0|000000|0$

due date $nnn|1111nn|1111nn||0001 \quad n \quad n|100000|0$



weight: $nnn|m000nn|m000nn||0000n - 2n - 3|0000n - 1n - 3|7$

processing time: $\underbrace{nnn}_{\text{vertex}} \ \underbrace{|1111nn|1111nn|}_{\text{large blocks}} \ \underbrace{|0000n - 2n - 3|0000n - 1n - 3|}_{\text{small blocks}} \ 0$

Small Blocks—(red, blue)-block

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 4 \cdot |E(H)|}$$

Now: Want to "count" edges $\{j, k\}$ with $(j, k) \leq (2, 3)$

For first edge $e = \{1, 3\}$, two jobs:

J^e with weight $|1000 \quad 0 \quad 0|000000|1$

processing time $|0001 \quad 0 \quad 0|000000|0$

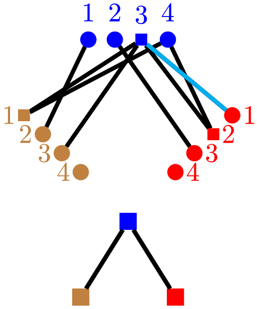
due date $nnn|1111nn|1111nn||0001n-2n-3|100000|0$

$\neg J^e$ with weight $|1000 \quad 0 \quad 0|000000|0$

processing time $|0001 \quad 0 \quad 0|000000|0$

due date $nnn|1111nn|1111nn||0001 \quad n \quad n|100000|0$

$\Rightarrow$ J^e can be early if $(n-1, n-3) \geq (n-2, n-3)$



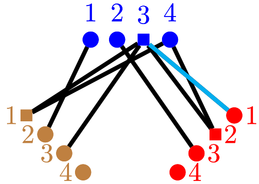
weight: $nnn|m000nn|m000nn||0000n-2n-3|0000n-1n-3|7$

processing time: $\underbrace{nnn}_{\text{vertex}} \underbrace{|1111nn|1111nn|}_{\text{large blocks}} \underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}0$

Small Blocks—(red, blue)-block

$$w_\# = 2 \cdot |V(H)| + 4 \cdot |E(H)|$$

Now: Want to "count" edges $\{j, k\}$ with $(j, k) \leq (2, 3)$



For first edge $e = \{1, 3\}$, two jobs:

J^e with weight $|1000 \quad 0 \quad 0|000000|1$

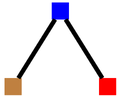
processing time $|0001 \quad 0 \quad 0|000000|0$

due date $nnn|1111nn|1111nn||0001n-2n-3|100000|0$

$\neg J^e$ with weight $|1000 \quad 0 \quad 0|000000|0$

processing time $|0001 \quad 0 \quad 0|000000|0$

due date $nnn|1111nn|1111nn||0001 \quad n \quad n|100000|0$



$\Rightarrow$ J^e can be early if $(n-1, n-3) \geq (n-2, n-3) \iff (1, 3) \leq (2, 3)$

weight: $nnn|m000nn|m000nn||0000n-2n-3|0000n-1n-3|7$

processing time: $\underbrace{nnn}_{\text{vertex}}\ \underbrace{|1111nn|1111nn|}_{\text{large blocks}}\ \underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}}\ 0$

Small Blocks—(red, blue)-block

$$w_{\#} = 2 \cdot |V(H)| + 4 \cdot |E(H)|$$

Now: Want to “count” edges $\{j, k\}$ with $(j, k) \leq (2, 3)$

For first edge $e = \{1, 3\}$, two jobs:

J^e with weight $|1000 \quad 0 \quad 0|000000|1$

processing time $|0001 \quad 0 \quad 0|000000|0$

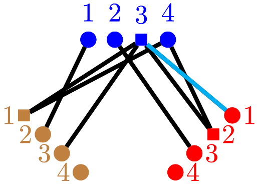
due date $nnn|1111nn|1111nn||0001n-2n-3|100000|0$

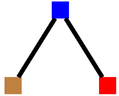
$\neg J^e$ with weight $|1000 \quad 0 \quad 0|000000|0$

processing time $|0001 \quad 0 \quad 0|000000|0$

due date $nnn|1111nn|1111nn||0001 \quad n \quad n|100000|0$

$\Rightarrow$ J^e can be early if $(n-1, n-3) \geq (n-2, n-3) \iff (1, 3) \leq (2, 3)$
one of J^e and $\neg J^e$ can be scheduled early.





weight: $nnn|m000nn|m000nn||0000n-2n-3|0000n-1n-3|7$

processing time: $\underbrace{nnn}_{\text{vertex}} \ \underbrace{|1111nn|1111nn|}_{\text{large blocks}} \ \underbrace{|0000n-2n-3|0000n-1n-3|}_{\text{small blocks}} \ 0$

After small blocks

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 8 \cdot |E(H)|}$$

Weight of early jobs: $\underbrace{nnn}_{\text{vertex}}\ \underbrace{|m000nn|m000nn|}_{\text{large blocks}}\ \underbrace{|m000nn|m000nn|}_{\text{small blocks}}\ \underbrace{|E(G)| + \ell}_{\text{counting}}$

where ℓ is the number of edges between the selected vertices

Processing time of early jobs $\underbrace{nnn}_{\text{vertex}}\ \underbrace{|1111nn|1111nn|}_{\text{large blocks}}\ \underbrace{|1111nn|1111nn|}_{\text{small blocks}}\ \underbrace{0}_{\text{counting}}$

After small blocks

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 8 \cdot |E(H)|}$$

Weight of early jobs: $\underbrace{nnn}_{\text{vertex}}\ \underbrace{|m000nn|m000nn|}_{\text{large blocks}}\ \underbrace{|m000nn|m000nn|}_{\text{small blocks}}\ \underbrace{|E(G)| + \ell}_{\text{counting}}$

where ℓ is the number of edges between the selected vertices

G contains H as colored subgraph $\iff$ $\ell \geq |E(H)|$

Processing time of early jobs $\underbrace{nnn}_{\text{vertex}}\ \underbrace{|1111nn|1111nn|}_{\text{large blocks}}\ \underbrace{|1111nn|1111nn|}_{\text{small blocks}}\ \underbrace{0}_{\text{counting}}$

After small blocks

$$\boxed{w_{\#} = 2 \cdot |V(H)| + 8 \cdot |E(H)|}$$

Weight of early jobs: $\underbrace{nnn}_{\text{vertex}}\ \underbrace{|m000nn|m000nn|}_{\text{large blocks}}\ \underbrace{|m000nn|m000nn|}_{\text{small blocks}}\ \underbrace{|E(G)|+\ell}_{\text{counting}}$

where ℓ is the number of edges between the selected vertices

G contains H as colored subgraph $\iff \ell \geq |E(H)|$

Theorem

Assuming ETH, there is no $n^{o(w_{\#}/\log w_{\#})}$-time algorithm for $1||\sum w_j U_j$.

Processing time of early jobs $\underbrace{nnn}_{\text{vertex}}\ \underbrace{|1111nn|1111nn|}_{\text{large blocks}}\ \underbrace{|1111nn|1111nn|}_{\text{small blocks}}\ \underbrace{0}_{\text{counting}}$

Conclusion

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Thank you!