The Longest Processing Time rule for identical parallel machines revisited

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Outline

ILP modeling and approximation

- Every standard undergraduate course on Operations Research (OR) embeds a section devoted to [Integer] Linear Programming (ILP) Modeling.

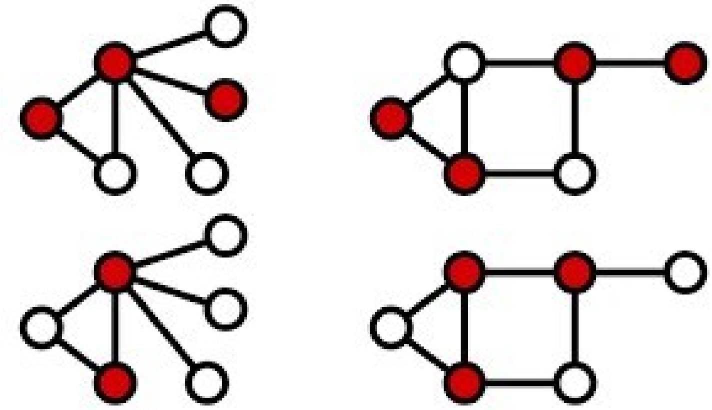
- OR experts and practitioners apply ILP models in order to
  - provide formal representations of real problems;
  - directly compute the corresponding solution by means of ILP solvers (unfortunately does not always work that well...);
  - compute heuristic solutions by means of matheuristics procedures embedding the solutions of ILP subproblems into local search approaches;
  - derive approximation bounds on problems where the related ILP formulations present strong structural properties

Approximation algorithms: standard notation

- OPT: optimal solution value
- A: solution value of the approximation algorithm
- $r_A = \frac{A}{OPT}$: performance ratio.
- We are typically interested in approximation algorithms requiring polynomial time complexity.

Approximation via **standard ILP modeling**: Minimum Vertex Cover

- Input: A graph $G = (V, E)$
- Definition: A vertex cover of G is a subset of V that covers (i.e., "touches") every edge in E.



Approximation via **standard ILP modeling**: Minimum Vertex Cover

- ILP formulation of the minimum vertex cover (MVC) problem

$$
\text{MVC} = \begin{cases} \min & \sum_{i \in V} x_i \\ & x_i + x_j \geqslant 1 \;\; \forall (i,j) \in E \\ & x_i \in \{0,1\} \;\; \forall i \in V \end{cases}
$$

$$
\text{MVC-R} = \begin{cases} \min & \sum_{i \in V} x_i \\ & x_i + x_j \geqslant 1 \;\; \forall (i,j) \in E \\ & 0 \leq x_i \leq 1 \;\; \forall i \in V \end{cases}
$$

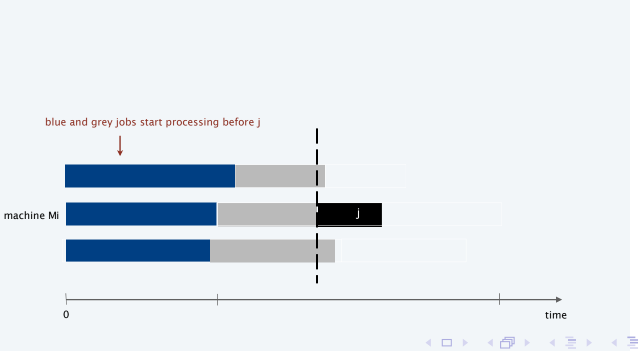
- Solving to optimality MVC-R (requires polynomial time) and setting $x_i = 1$ for all variables with value ≥ 0.5 provides a 2-approximation ratio.

Approximation via **non standard ILP modeling**

- We focus here on **non standard ILP modeling** for approximation.
- The aim is to mimick by ILP modeling the behavior of a procedure (typically greedy).
- We apply this approach to
    - Machine Scheduling: problem $P||C_{\max}$ and the Longest Processing Time rule.

Parallel machines scheduling: Introduction

- We consider problem $P_m||C_{max}$ where the goal is to schedule n jobs on m identical parallel machines M_i $(i = 1, \ldots, m)$ minimizing the makespan.
- We revisit the famous Longest Processing Time (LPT) rule proposed by Graham - 1969:
  sort the jobs $1, ..., n$ in non-ascending order of their processing times p_j $(j = 1, \ldots, n)$ and then assign one job at a time to the machine whose load is smallest so far.



Parallel machines scheduling: Introduction

- Assume the jobs indexed by non-increasing p_j ($p_j \geq p_{j+1}$, $j = 1, \ldots, n-1$).
- Denote the solution values of the LPT schedule and the optimal makespan by C_m^{LPT} and C_m^* respectively, where index m indicates the number of machines.
- Denote by $r_k = \frac{C_m^{LPT}}{C_m^*}$ the performance ratio of the LPT schedule when k jobs are assigned to the machine yielding the maximum completion time (the critical machine).
- Denote by j' denotes the critical job (the job inducing the makespan).

$P_m||C_{max}$ problem and LPT rule properties

- $C^*_m \geq \max\{p_1, \frac{\sum_{j=1}^{n} p_j}{m}\}$.
- C^{LPT}_m is optimal if $p_{j'} > \frac{C^*_m}{3}$.
- $C^{LPT}_m \leq \frac{\sum_{j=1}^{j'-1} p_j}{m} + p_{j'} \leq C^*_m + p_{j'}(1 - \frac{1}{m}) \leq (\frac{4}{3} - \frac{1}{3m})C^*_m$
  [Graham 1969 - see also https://elementsofscheduling.nl chap. 8].
- For each job i assigned by LPT to position j on a machine:
  $p_i \leq \frac{C^*_m}{j}$
  [Chen 1993].

LPT rule properties:

Known LPT performance ratios.

- $r_1 = 1$.
- $r_2 \leq \frac{4}{3} - \frac{1}{3(m-1)}$ - [Chen 1993].
- $r_3 \leq \frac{4}{3} - \frac{1}{3m}$ - [Graham 1969].
- $r_k \leq \frac{k+1}{k} - \frac{1}{km} \quad k \geq 3$ [Coffman and Sethi 1976 - generalizes Graham].

Notice that

- $r_2 = 1$ for $m = 2$;
- $r_2 = r_4$ for $m = 3$, $r_2 < r_4$ for $m \geq 4$;
- $r_k < r_{k+1}$ for $k \geq 3$

$\Longrightarrow$ Improving r_3 improves LPT.
We focus then on instances where the critical job is in position 3.

Tight worst-case examples for LPT

- 2 machines - 5 jobs $\longrightarrow [3, 3, 2, 2, 2]$.
- $C^*_{m=2} = 6, \qquad C^{LPT}_{m=2} = 7, \qquad r_3 = \frac{4}{3} - \frac{1}{3m} = \frac{7}{6}$.
- 3 machines, 7 jobs $\longrightarrow [5, 5, 4, 4, 3, 3, 3]$.
- $C^*_{m=3} = 9, \qquad C^{LPT}_{m=3} = 11, \qquad r_3 = \frac{4}{3} - \frac{1}{3m} = \frac{11}{9}$.
- m machines, $2m + 1$ jobs
  $\longrightarrow [2m - 1, 2m - 1, 2m - 2, 2m - 2, ..., m, m, m]$.
- $C^*_m = 3m = \frac{\sum_{i=1}^n p_i}{m}, \qquad C^{LPT}_m = 4m - 1,$
  $r_3 = \frac{4m-1}{3m} = \frac{4}{3} - \frac{1}{3m}$.
- Worst-case always occurs with $2m + 1 = n$ jobs where the critical job is job n in position 3 and when $C^*_m = \sum_{i=1}^n p_i/m$.

LPT revisited

- We assume that the critical job in *LPT* is the last one, namely $j' = n$. If not, we would have further jobs after the critical job that do not affect the makespan provided by *LPT* but can contribute to increasing the optimal solution value.

- We analyze for $m \geq 3$:
  - $2m + 2 \leq n \leq 3m$ (or else the critical job would be in position ≥ 4);
  - $n = 2m + 1$.

- We recall that for $n \leq 2m$ the performance ratio of *LPT* is $\leq \frac{4}{3} - \frac{1}{3(m-1)}$.

- We employ **Linear Programming** to perform the analysis.

LPT revisited: $2m + 2 \leq n \leq 3m$

Proposition

If LPT schedules at least 3 jobs on a non crit. machine before assigning the crit. job, then LPT has an approx. bound $\leq \frac{4}{3} - \frac{1}{3(m-1)}$ *for* $m \geq 5$.

Sketch of proof.

- We assume n in position 3, or else either r_2 holds or at least r_4 holds. Hence, LPT schedules at least another job in position ≥ 3.
- We consider an LP model where we arbitrarily set the value C_m^{LPT} to 1 and minimize the value of C_m^*.
- L denotes the starting time of job n, i.e. $C_m^{LPT} = L + p_n$.
- C_1 denotes the compl. time of the non-crit. machine processing at least 3 jobs.
- C_2 denotes the sum of compl. times of the other $(m-2)$ machines, i.e. $C_2 = \sum_{j=1}^{n} p_j - C_1 - (L + p_n)$.
- Due to list scheduling, condition $\frac{C_2}{m-2} \geq L$ holds.
- As n is in position 3, condition $p_n \leq \frac{C_m^*}{3}$ holds.

LPT revisited: $2m+2 \leq n \leq 3m$ (LP formulation)

- We associate non-negative variables p_n and $sump$ with p_n and $\sum_{j=1}^{n} p_j$.
- We associate non-negative variables c_1, c_2, l, opt with C_1, C_2, L and C_m^*.
- The following LP model (for given m) holds:

$$
\begin{aligned}
\text{minimize} \quad & opt && (1)\\
& -m \cdot opt + sump \leq 0 && (2)\\
& 3 \cdot p_n - c_1 \leq 0 && (3)\\
& l - c_1 \leq 0 && (4)\\
& (m-2)l - c_2 \leq 0 && (5)\\
& c_1 + l + p_n + c_2 - sump = 0 && (6)\\
& l + p_n = 1 && (7)\\
& p_n - \frac{opt}{3} \leq 0 && (8)\\
& p_n, sump, c_1, c_2, l, opt \geq 0 && (9)
\end{aligned}
$$

LPT revisited: $2m + 2 \leq n \leq 3m$ (LP formulation)

- The minimization of the objective function (1), after setting w.l.o.g LPT solution value to 1 (constraint (7)), provides an upper bound on the performance ratio of the LPT rule.
- Constraint (2): $-m \cdot opt + sump \leq 0$ corresponds to bound $C_m^* \geq \frac{\sum_{j=1}^{n} p_j}{m}$.
- Constraint (3): $3 \cdot p_n - c_1 \leq 0$ states that the value of c_1 is at the least $3p_n$, since 3 jobs with proc. time $\geq p_n$ are assigned to a non critical machine.
- Constraint (4): $l - c_1 \leq 0$ states that the compl. time of the critical machine before the last job is loaded is less than the compl. time of the other machine processing at least three jobs.
- Constraint (5): $(m-2)l - c_2 \leq 0$ fulfills the list scheduling requirement.
- Constraint (6): $c_1 + l + p_n + c_2 - sump = 0$ guarantees that variable $sump$ represents $\sum_{j=1}^{n} p_j$
- Constraint (8) corresponds to condition $p_n \leq \frac{C_m^*}{3}$.
- Constraints (9) state that all variables are non-negative.

LPT revisited: $2m + 2 \leq n \leq 3m$ (LP formulation)

- The proposed LP model is continuous and contains just 6 variables and 7 constraints for any fixed m.
- By strong duality (and a little bit of reverse engineering) it is possible to show that in the optimal solution, for any $m \geq 5$, the variables values are as follows

$$p_n = \frac{m-1}{4m-5}; \qquad sump = \frac{3m(m-1)}{4m-5};$$

$$c_1 = \frac{3(m-1)}{4m-5}; \qquad c_2 = \frac{(m-2)(3(m-1)-1)}{4m-5};$$

$$l = \frac{3(m-1)-1}{4m-5}; \qquad opt = \frac{3(m-1)}{4m-5}.$$

- Correspondingly, we have $\frac{C_m^{LPT}}{C_m^*} \leq 1/opt = \frac{4m-5}{3(m-1)} = \frac{4}{3} - \frac{1}{3(m-1)}$.
- Notice that this bound is not tight.

LPT revisited: other subcases

With similar analysis, and sometimes partial enumeration, the following propositions also hold.

- For $3 \leq m \leq 4$ and $2m+2 \leq n \leq 3m$, LPT (with job n critical) has an approximation ratio $\leq \frac{4}{3} - \frac{1}{3(m-1)}$ for $3 \leq m \leq 4$.
- For $n \leq 2m$ and $m \geq 3$, LPT has an approximation ratio $\leq \left(\frac{4}{3} - \frac{1}{3(m-1)}\right)$.
- For $m \geq 3$ and $n = 2m+1$, if LPT loads at least three jobs on a machine before the critical job, then it has an approximation ratio $\leq \left(\frac{4}{3} - \frac{1}{3(m-1)}\right)$.

The only case remaining is then related to instances with $n = 2m+1$ where LPT schedules job n only in third position and n is critical.

Improving LPT

- Consider a slight algorithmic variation where a set of the sorted jobs is first loaded on a machine and then LPT is applied on the remaining job set.
- Let denote this variant as $LPT(\mathcal{S})$ where $\mathcal{S}$ represents the set of jobs assigned all together to a machine first.

We consider the following Algorithm 1.

Input: $P_m||C_{max}$ instance with n jobs and $m \geq 3$ machines.
- Apply LPT yielding a schedule with makespan z_1.
- Apply $LPT' = LPT(\{j'\})$ with solution value z_2.
- Return $\min\{z_1, z_2\}$.

In practice, this algorithm applies LPT first and then re-applies LPT *after* having loaded first on a machine its critical job j'.

Improving LPT

Whenever both LPT and LPT' are applied, the following subcases need to be considered

1. $n = j' = 2m + 1$ with subcases
   1. $p_{2m+1} \geq p_1 - p_m$.
   2. $p_{2m+1} < p_1 - p_m$.
2. $n \geq i > j' = 2m + 1$ with i critical in LPT'.
3. $n > j' = 2m + 1 \geq i$ with i critical in LPT'.

We focus here on subcase 1.1 ($n = j' = 2m + 1$ and $p_{2m+1} \geq p_1 - p_m$): all other subcases provide with similar analysis approximation ratio not superior to $\frac{4}{3} - \frac{1}{3(m-1)}$.

Handling instances with $n = j' = 2m + 1$ and $p_{2m+1} \geq p_1 - p_m$

- Note that LPT must couple jobs $1, \ldots, m$ respectively with jobs $2m, \ldots, m+1$ on the m machines before scheduling job $2m+1$, or else LPT has an approximation ratio $\leq \left(\frac{4}{3} - \frac{1}{3(m-1)}\right)$.
- Hence, the LPT schedule is as follows -
  we denote by $C(M_i)$ the completion time of machine M_i

$$
\begin{aligned}
&M_1 : 1, 2m \quad \rightarrow C(M_1) = p_1 + p_{2m} \\
&M_2 : 2, 2m-1 \quad \rightarrow C(M_2) = p_2 + p_{2m-1} \\
&\ldots \\
&M_{m-1} : m-1, m+2 \quad \rightarrow C(M_{m-1}) = p_{m-1} + p_{m+2} \\
&M_m : m, m+1 \quad \rightarrow C(M_m) = p_m + p_{m+1}
\end{aligned}
$$

where job $2m+1$ will be assigned to the machine with minimum completion time.

Handling instances with $n = j' = 2m+1$ and $p_{2m+1} \geq p_1 - p_m$

LPT' can be shown to be as follows

$$
\begin{aligned}
&M_1 : 2m+1, m, 2m \quad \rightarrow C(M_1) = p_{2m+1} + p_m + p_{2m} \\
&M_2 : 1, 2m-1 \quad \rightarrow C(M_2) = p_1 + p_{2m-1} \\
&M_3 : 2, 2m-2 \quad \rightarrow C(M_3) = p_2 + p_{2m-2} \\
&\ldots \\
&M_{m-1} : m-2, m+2 \quad \rightarrow C(M_{m-1}) = p_{m-2} + p_{m+2} \\
&M_m : m-1, m+1 \quad \rightarrow C(M_m) = p_{m-1} + p_{m+1}
\end{aligned}
$$

with subcases

1. LPT' makespan is on M_1.
2. LPT' makespan is on $M_2, \ldots M_m$.

We focus here on subcase 1 (LPT' makespan is on M_1): the other subcase provides with similar analysis approximation ratio not superior to $\frac{4}{3} - \frac{1}{3(m-1)}$.

Case $n = j' = 2m+1$, $p_{2m+1} \geq p_1 - p_m$, LPT' makespan on M_1

- If LPT' is not optimal, then it can be shown that $C_m^* \geq p_{m-1} + p_m$.
- We get the following result.

Proposition

If $p_{2m+1} \geq p_1 - p_m$ and LPT' makespan is equal to $p_{2m+1} + p_m + p_{2m}$, then the proposed algorithm has an approximation ratio not superior to $\frac{7}{6}$.

- Proof: again we employ Linear Programming to evaluate the performance of LPT'. We consider non-negative variables x_j associated with p_j $(j = 1, \ldots, n)$ and a positive parameter $OPT > 0$ associated with C_m^*.

Case $n = j' = 2m + 1$, $p_{2m+1} \geq p_1 - p_m$, LPT' makespan on M_1

The LP model is

$$\begin{aligned}
\text{maximize} \quad & x_{2m+1} + x_m + x_{2m} && (10)\\
\text{subject to} \quad & x_{m-1} + x_m \leq OPT && (11)\\
& x_{2m-1} + x_{2m} + x_{2m+1} \leq OPT && (12)\\
& x_{2m+1} - (x_1 - x_m) \geq 0 && (13)\\
& x_1 - x_{m-1} \geq 0 && (14)\\
& x_{m-1} - x_m \geq 0 && (15)\\
& x_m - x_{m+1} \geq 0 && (16)\\
& x_{m+1} - x_{2m-1} \geq 0 && (17)\\
& x_{2m-1} - x_{2m} \geq 0 && (18)\\
& x_{2m} - x_{2m+1} \geq 0 && (19)\\
& x_1, x_{m-1}, x_m, x_{m+1}, x_{2m-1}, x_{2m}, x_{2m+1} \geq 0 && (20)
\end{aligned}$$

Case $n = j' = 2m + 1$, $p_{2m+1} \geq p_1 - p_m$, LPT' makespan on M_1

- The objective function value (10) represents an upper bound on the worst case performance of the algorithm.
- Constraints (11)–(12) correspond to $C^*_m \geq p_{m-1} + p_m$ and $C^*_m \geq p_{2m-1} + p_{2m} + p_{2m+1}$.
- Constraint (13) corresponds to the initial assumption $p_{2m+1} \geq p_1 - p_m$.
- Constraints (14)–(19) state that the considered relevant jobs are sorted by non-increasing processing times.
- Constraints (20) indicate that the variables are non-negative.
- Further viable constraints where not necessary to reach the required result. By setting $OPT = 1$, the cost function has value $\frac{7}{6}$.

Putting things together, the following theorem holds

Theorem

The proposed algorithm has an approximation ratio not superior to $\frac{4}{3} - \frac{1}{3(m-1)}$ *for* $m \geq 3$.

From approximation to heuristics

- W.r.t. the worst-case analysis for $m \geq 3$, the relevant subcase was the one with $p_{2m+1} \geq p_1 - p_m$ and LPT' required to schedule p_{2m+1} initially and then apply list scheduling first to the sorted jobset $p_1, ..., p_m$ according to LPT and then to the sorted jobset $p_{m+1}, ..., p_{2m}$ always according to LPT.

- We propose then an alternative approach that splits the sorted job set in tuples of m consecutive jobs $(1, \ldots, m; m+1, \ldots, 2m;$ etc.) and sorts the tuples in non-increasing order of the difference between the largest job and the smallest job in the tuple. Then a list scheduling is applied to the set of sorted tuples. We denote this approach as $SLACK$.

From approximation to heuristics

The $SLACK$ heuristic:

Input: $P_m||C_{max}$ instance m machines and n jobs with processing times p_j $(j = 1, \ldots, n)$.
- Sort items by non-increasing p_j.
- Consider tuples $1, \ldots, m$; $m+1, \ldots, 2m; \ldots; n-m+1, \ldots, n$ (if n/m is not integer, add dummy jobs with null proc. time in the last tuple).
- For each tuple, compute the associated slack
$p_1 - p_m$; $p_{(m+1)} - p_{2m}; \ldots; p_{(n-m+1)} - p_n$.
- Sort tuples by non-increasing slack.
- Apply List Scheduling to this job ordering and return the solution.

SLACK heuristic

- Worst-case example for LPT with 3 machines, 7 jobs $\longrightarrow [5, 5, 4, 4, 3, 3, 3]$.
- By adding 2 dummy jobs, we get $[5, 5, 4, 4, 3, 3, 3, 0, 0]$ that is $[5, 5, 4], [4, 3, 3], [3, 0, 0]$.
- Sorting tuples by non-increasing slack, we have $[3, 0, 0], [5, 5, 4], [4, 3, 3]$.
- Applying list scheduling, we get $C(1) = 10,\ C(2) = 9,\ C(3) = 8$ hence $C_{\max}^{SLACK} = 10$.

Since the construction and sorting of the tuples can be performed in $\mathcal{O}(n + m \log m)$, the running time of SLACK is $\mathcal{O}(n \log n)$ due to the initial jobs LPT sorting.

Computational testing

We compared *SLACK* to *LPT* on benchmark literature instances (Iori, Martello 2008)

- Two classical classes of instances from literature are considered: *uniform instances* (França et al. 1994) and *non-uniform instances* (Frangioni et al. 2004).
- In *uniform instances* the processing times are integer uniformly distributed in the range $[a, b]$. In *non-uniform instances*, 98% of the processing times are integer uniformly distributed in $[0.9(b-a), b]$ while the remaining ones are uniformly distributed in $[a, 0.2(b-a)]$. For both classes, we have $a = 1; b = 100, 1000, 10000$.
- For each class, the following values were considered for the number of machines and jobs: $m = 5, 10, 25$ and $n = 10, 50, 100, 500, 1000$.
- For each pair (m, n) with $m < n$, 10 instances were generated for a total of 780 instances.

Computational testing

			SLACK wins		draws		*LPT* wins	
$[a, b]$	m	Instances	#	(%)	#	(%)	#	(%)
1-100	5	50	31	(62.0)	16	(32.0)	3	(6.0)
	10	40	32	(80.0)	8	(20.0)	0	(0.0)
	25	40	23	(57.5)	17	(42.5)	0	(0.0)
1-1000	5	50	39	(78.0)	10	(20.0)	1	(2.0)
	10	40	40	(100.0)	0	(0.0)	0	(0.0)
	25	40	27	(67.5)	12	(30.0)	1	(2.5)
1-10000	5	50	39	(78.0)	10	(20.0)	1	(2.0)
	10	40	40	(100.0)	0	(0.0)	0	(0.0)
	25	40	28	(70.0)	10	(25.0)	2	(5.0)
Overall			299	(76.7)	83	(21.3)	8	(2.0)

Table: $P_m||C_{max}$ non uniform instances.

Computational testing

			SLACK wins		draws		*LPT* wins	
$[a, b]$	m	Instances	#	(%)	#	(%)	#	(%)
1-100	5	50	12	(24.0)	37	(74.0)	1	(2.0)
	10	40	14	(35.0)	20	(50.0)	6	(15.0)
	25	40	10	(25.0)	29	(72.5)	1	(2.5)
1-1000	5	50	32	(64.0)	15	(30.0)	3	(6.0)
	10	40	27	(67.5)	5	(12.5)	8	(20.0)
	25	40	24	(60.0)	12	(30.0)	4	(10.0)
1-10000	5	50	36	(72.0)	12	(24.0)	2	(4.0)
	10	40	37	(92.5)	0	(0.0)	3	(7.5)
	25	40	22	(55.0)	11	(27.5)	7	(17.5)
Overall			214	(54.9)	141	(36.1)	35	(9.0)

Table: $P_m||C_{max}$ uniform instances.

Computational testing

- $SLACK$ shows up to be clearly superior to LPT: on 780 benchmark literature instances, $SLACK$ wins 513 times, ties 224 times and loses 43 times only.

- $SLACK$ shows up to be competitive also to other similar state-of-the-art heuristics. It it is clearly superior to $COMBINE$ (Lee and Massey 1988) while it is slightly inferior to LDM (Karmarkar and Karp 1982) though being more than an order of magnitude faster.
- By adding a simple neighborhood search (NS) procedure in cascade, SLACK + NS becomes already superior to LDM still being more than an order of magnitude faster.

Conclusions

- We discussed how non standard ILP modeling can be successfully applied to derive improved approximation results.
- We considered problem $P_m||C_{max}$ and revisited the LPT rule.
- By means of Linear Programming we improved Graham's bound from $\frac{4}{3} - \frac{1}{3m}$ to $\frac{4}{3} - \frac{1}{3(m-1)}$ for $m \geq 3$.
- By similar analysis, a linear time algorithm for problem $P2||C_{\max}$ with a 13/12 approximation ratio can be derived;
- From the approximation analysis, we derived a simple $O(n \log n)$ heuristic procedure that drastically improves upon the performances of LPT.
- We believe that the proposed LP-based analysis can be successfully applied in approximation theory as a valid alternative to formal proof systems based on analytical derivation.

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