A Subexponential Time Algorithm for Makespan Scheduling of Unit Jobs with Precedence Constraints

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Saarland University and Max Planck Institute for Informatics

$P3|prec, p_j = 1|C_{\max}$

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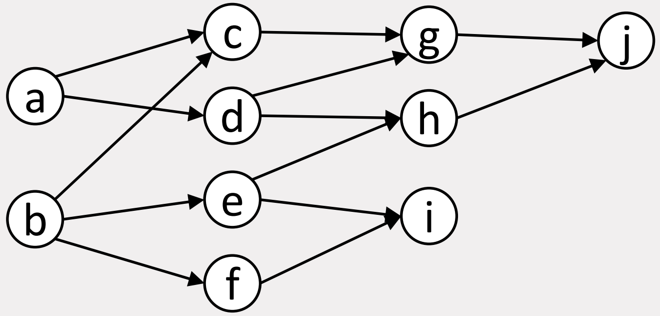
3 identical
parallel machines

$P3|prec, p_j = 1|C_{\max}$

Given:

- n jobs of length 1

$P3|prec, p_j = 1|C_{\max}$



Given:

- n jobs of length 1
- A precedence graph G

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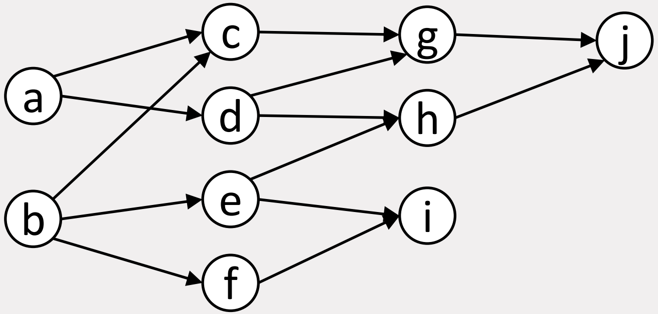

Given:

- n jobs of length 1
- A precedence graph G
- $T \in \mathbb{N}$

Q: Is there a schedule of makespan T?

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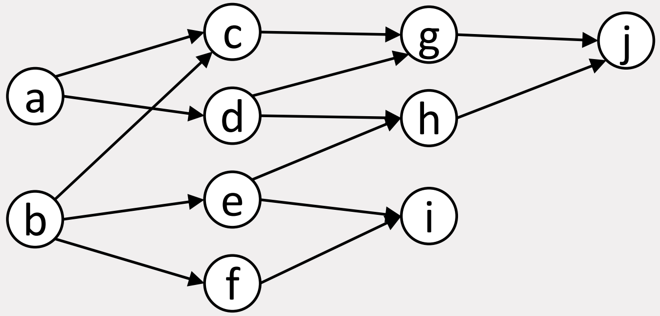

time →

	1	2	3	4
1	a	c	f	i
2	b	d	g	j
3		e	h	

$P3|prec, p_j = 1|C_{\max}$

Given:

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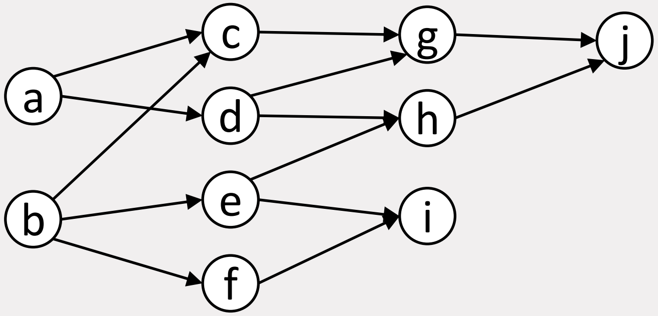
- $T \in \mathbb{N}$

G defines the problem

Q: Is there a schedule of makespan T?



time

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Observation:
Jobs of length one ⇒ `timeslots’

TU/e

List of Open Problems by Garey and Johnson 1979

1. Graph Isomorphism
2. Subgraph Homeomorphism
3. Graph genus
4. Chordal graph completion
5. Chromatic index
6. Spanning tree parity problem
7. Partial order dimension
8. Precedence constrained 3-processor scheduling
9. Linear Programming
10. Total unimodularity
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   $2^{O\left((\log n)^3\right)}$ time
   [Babai 2017]

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$2^{O\left(\sqrt{n}\cdot\log n\right)}$ time
This talk

Why Focus on Subexponential?

- Typically for NP-complete problems with
  - geometrical properties *(planar graph, Euclidean settings)*
  - Parameters > number of vertices *(edge deletion to …)*

- Stepping stone towards (quasi)-polynomial

  (e.g. Parity Games, Independent Set on Pk-free graphs)

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Before:

$Pm|prec, p_j = 1|C_{\max}$ can be solved in $O\left(2^n \cdot \binom{n}{m}\right)$ time.

Our Result

Our result:

$Pm|prec, p_j = 1|C_{\max}$ can be solved in $\left(1 + \frac{n}{m}\right)^{O(\sqrt{nm})}$ time.

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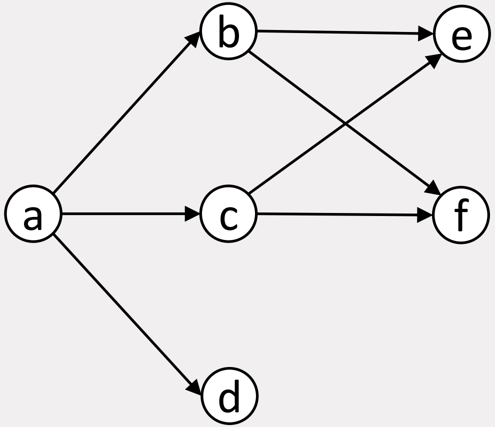
Two ways to explain, but main insights:

1. Use of look-up table
2. Keeping track of number of isolated vertices
3. Finding win-win strategy

Definitions

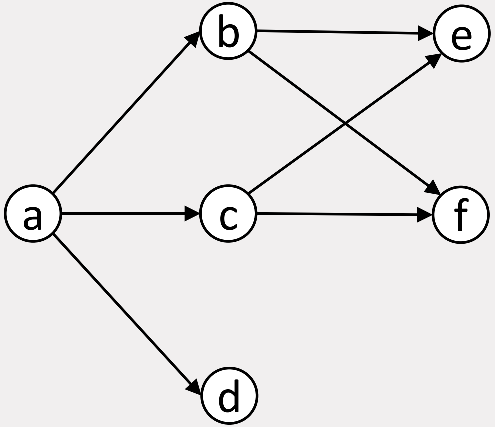


Precedence Constraints Graph G

Definitions

$G \Rightarrow$ partial order:

- $i \prec j$ if $(i, j) \in G$



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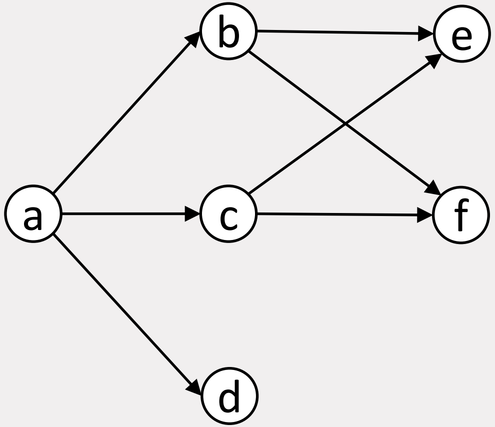

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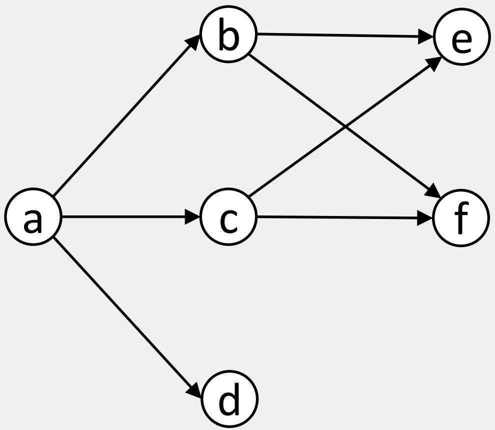

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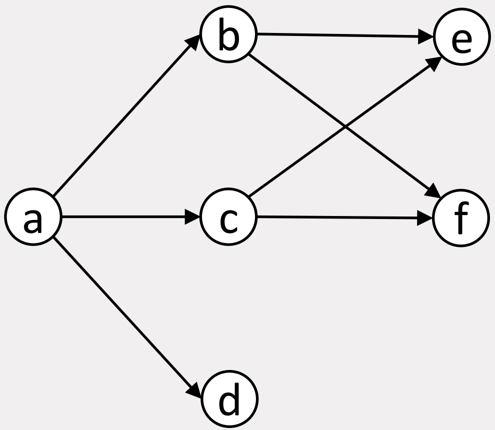

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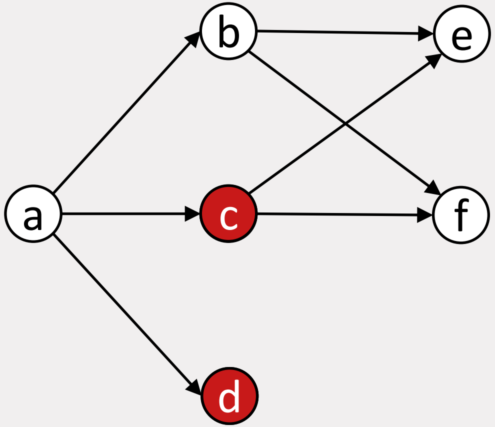

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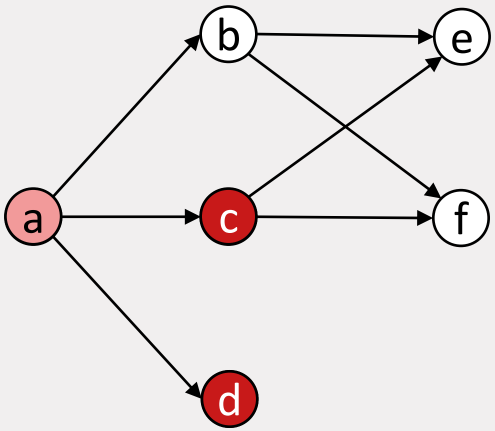

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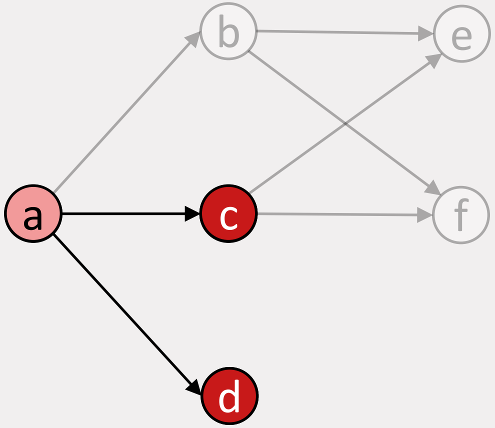

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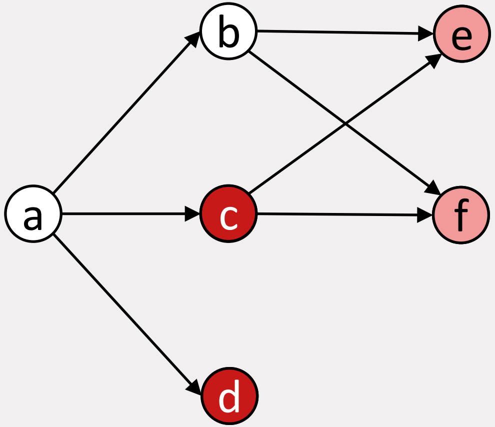

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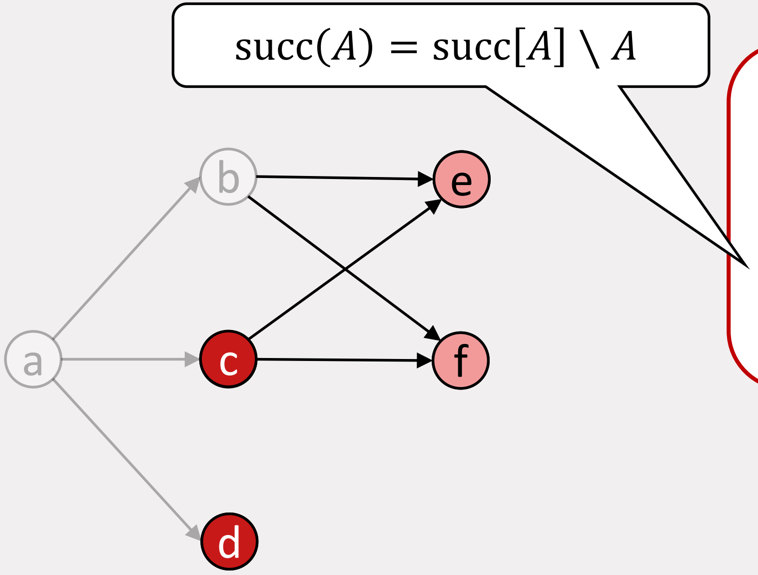

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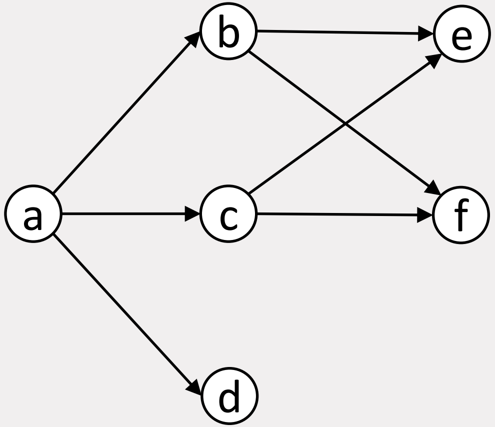
Definitions

Def: A *chain* is a set A whose elements are pairwise **comparable**.



Precedence Constraints Graph G

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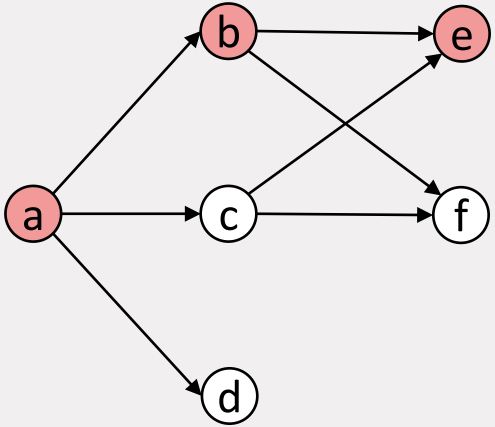
Def: A *chain* is a set A whose elements are pairwise **comparable**.
Def: The *height* $h(G)$ is the size of the longest chain (in #arcs).

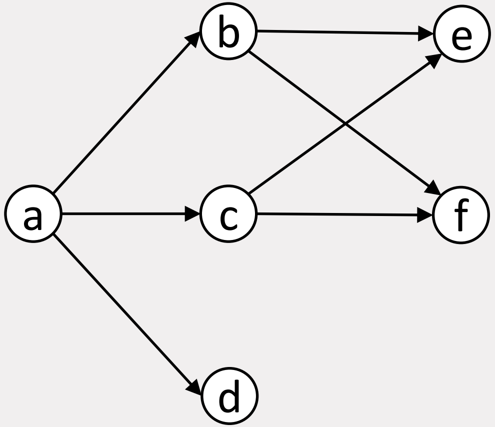
In the example $h(G) = 2$



Precedence Constraints Graph G

If $h(G) = 0$, then there are no arcs

Definitions



Precedence Constraints Graph G

Def: A *chain* is a set A whose elements are pairwise **comparable**.
Def: The *height* $h(G)$ is the size of the longest chain (in #arcs).

In the example $h(G) = 2$

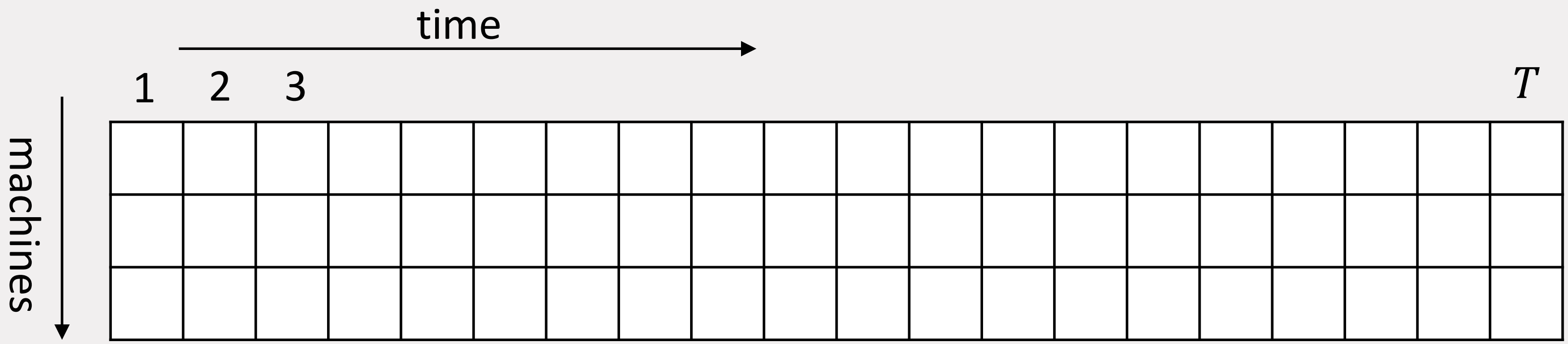
Def: An *antichain* is a set A whose elements are pairwise **incomparable**.

Examples of antichains in G

- ✓ $\{b, c, d\}$
- ✓ $\{b, c\}$
- ✓ $\{d, f\}$

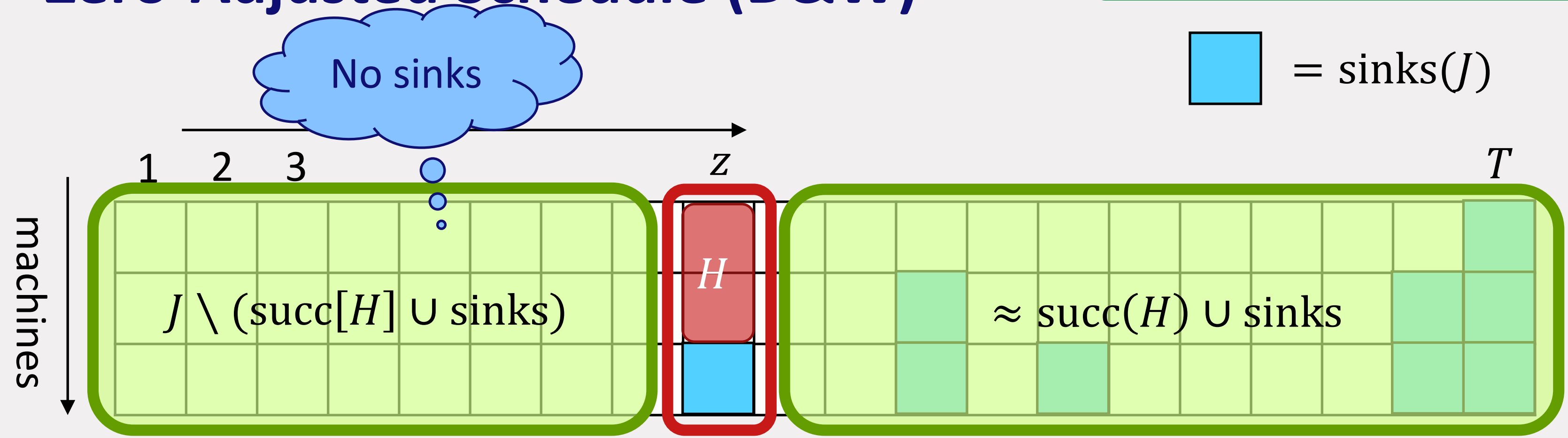
Jobs in one timeslot always form an antichain

Zero-Adjusted Schedule (D&W)



Zero-Adjusted Schedule (D&W)

Assumption: $n = 3 \cdot T$



Let $z \in [1, T]$ be the first moment with a sink.

D&W: W.m.a. Each job x **after *z*** is a sink or a successor of a job at time z.

Dolev and Warmuth

Recursive

Schedule(J):

1. **if** $h(G[J]) = 0$ (i.e. $\text{sinks}(J) = J$) **return** $\left\lceil \frac{|J|}{3} \right\rceil$
2. **else return** $\min_{H \in \text{Sep}(J)} \{\text{Schedule}(\text{left}(J,H)) + \text{Schedule}(\text{right}(J,H)) + 1\}$

$\text{Sep}(J) := \{ H \subseteq J$ s.t.
(1) $|H| \leq 3$,
(2) H is antichain,
(3) $|H \setminus \text{sinks}(J)| < 3\}$

$\text{left}(J,H) := J \setminus (\text{succ}[\text{H}] \cup \text{sinks}(J))$
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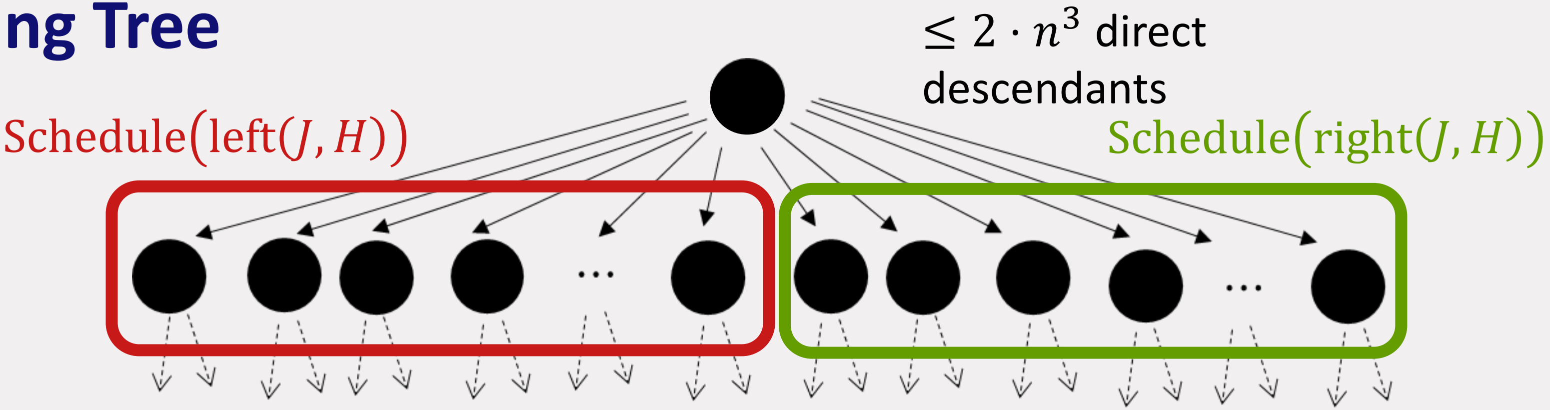
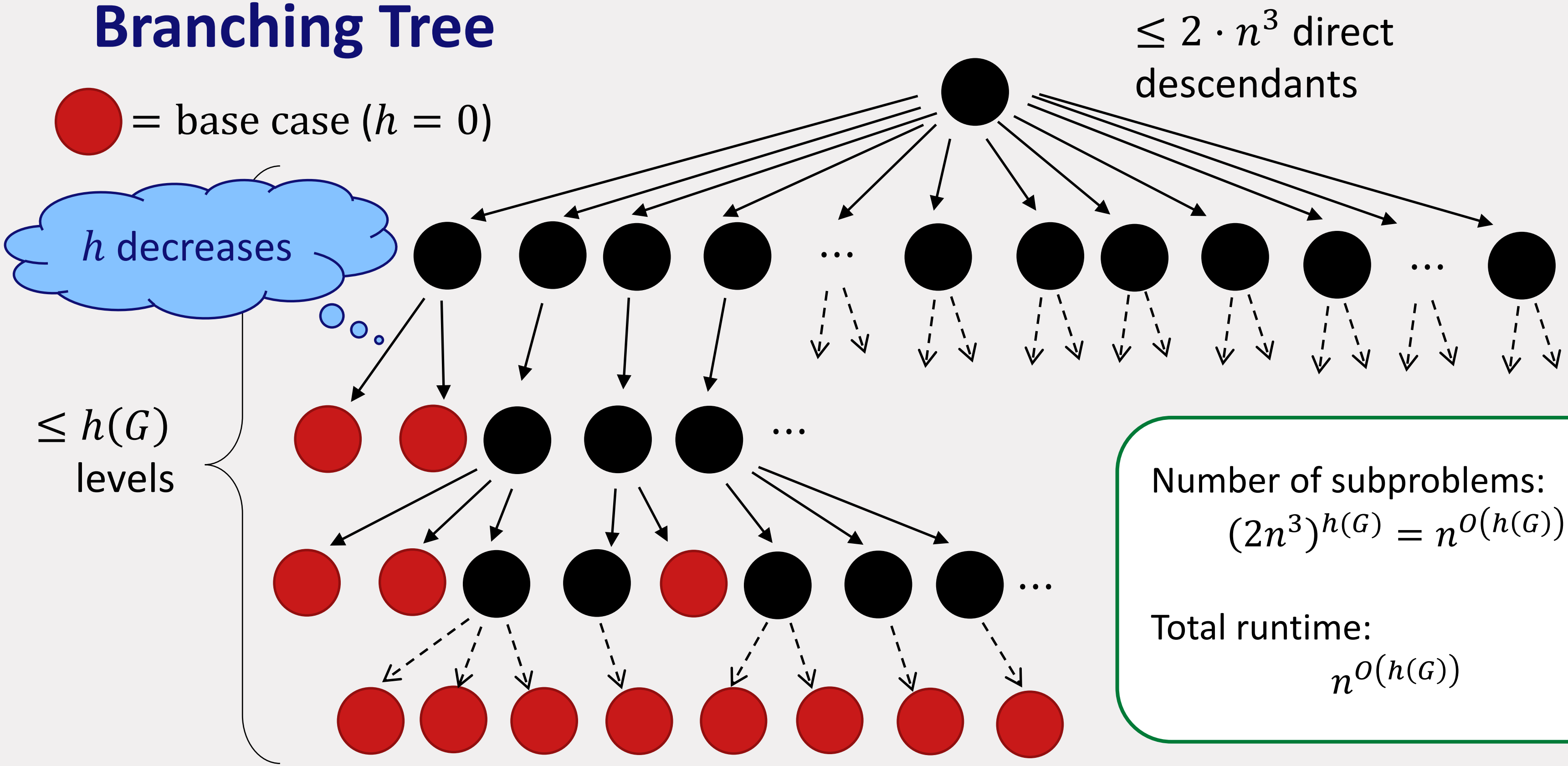
Each subproblem: height decreases by ≥ 1!

Branching Tree



Branching Tree



Number of subproblems:

$$(2n^3)^{h(G)} = n^{O(h(G))}$$

Total runtime:

$$n^{O(h(G))}$$

D&W

$\mathrm{Schedule}(J)$:

1. **if** $h(G[J]) = 0$ **return** $\left\lceil \frac{|J|}{3} \right\rceil$
2. **for each** $H \in \mathrm{Sep}(J)$ **do:**
3. $\quad\quad \mathrm{OPT}[\mathrm{left}(J,H)] \coloneqq \mathrm{Schedule}\big(\mathrm{left}(J,H)\big)$
4. $\quad\quad \mathrm{OPT}[\mathrm{right}(J,H)] \coloneqq \mathrm{Schedule}(\mathrm{right}(J,H))$
5. $\mathrm{OPT}[J] := \min_{H \in \mathrm{Sep}(J)} \{\mathrm{OPT}[\mathrm{left}(J,H)] + \mathrm{OPT}[\mathrm{right}(J,H)] + 1\}$

6. Return $\mathrm{OPT}[J]$

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D&W + LookUp Table

Schedule(J):

1. **return** LUT[J] if it was already set
2. **if** $h(G[J]) = 0$ **return** $\left\lceil \frac{|J|}{3} \right\rceil$
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6. $\text{OPT}[J] := \min_{H \in \text{Sep}(J)} \{\text{OPT}[\text{left}(J,H)] + \text{OPT}[\text{right}(J,H)] + 1\}$
7. $\text{LUT}[J] = \text{OPT}[J]$
8. Return $\text{OPT}[J]$

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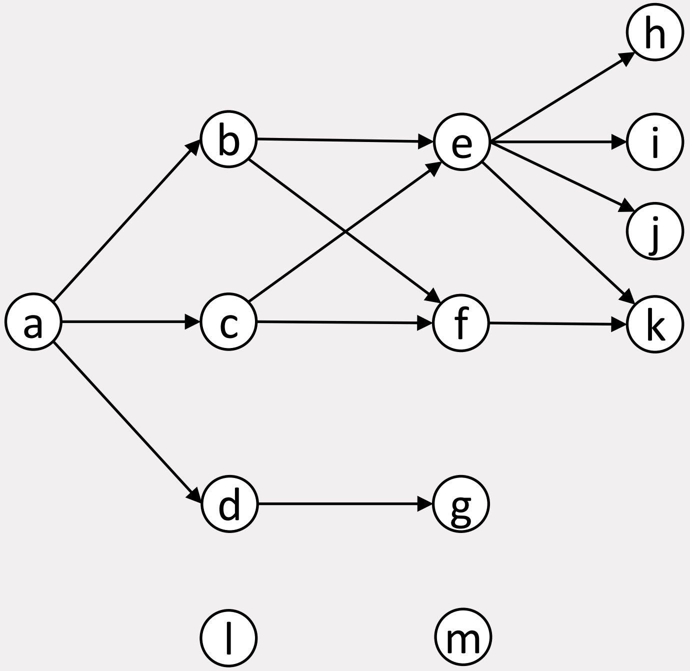
Too many different problems!

What to store?



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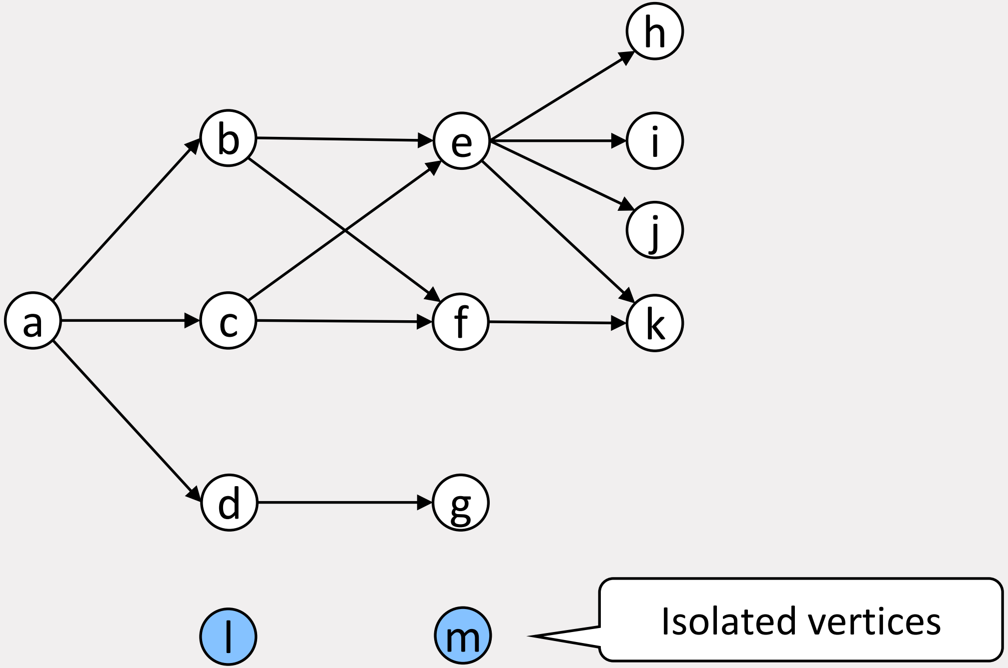

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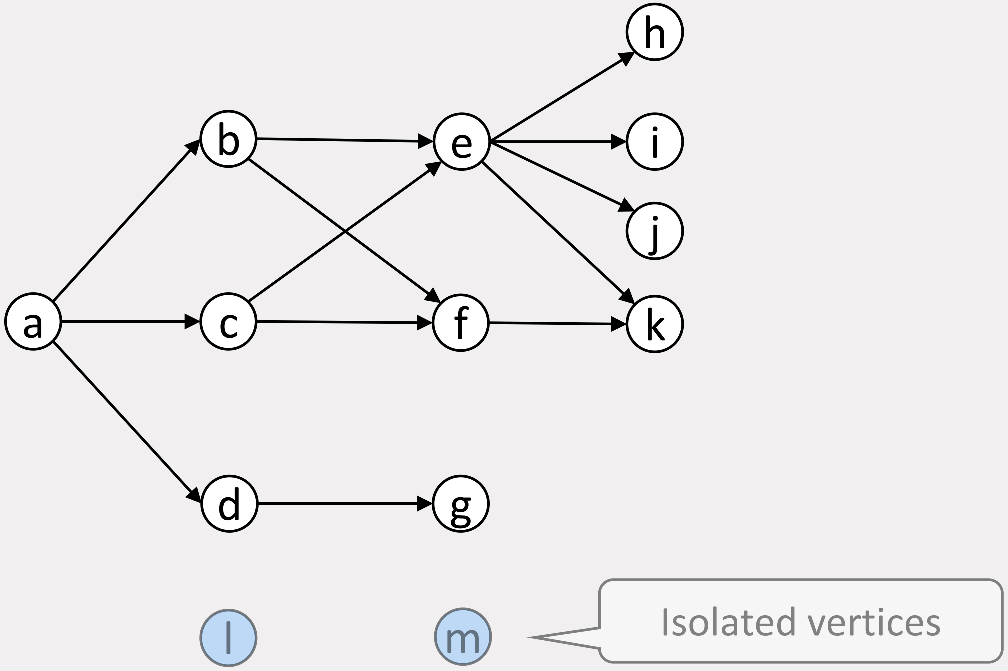

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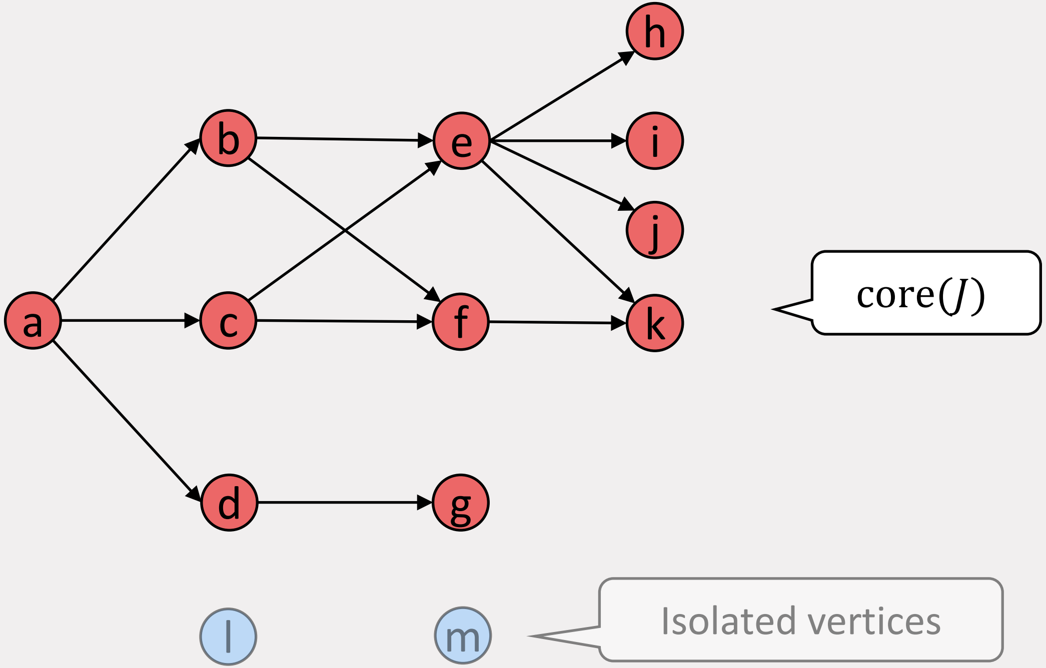

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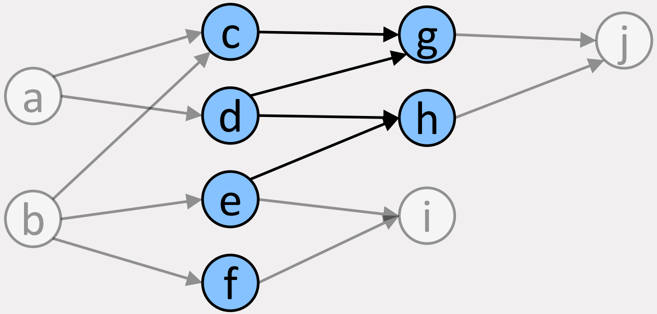
$\mathrm{Schedule}(J)$:

1. **return** $\mathrm{LUT}[\mathrm{core}(J), \#\mathrm{iso}(J)]$ if it was already set
2. **if** $J = \emptyset$ **return** 0
3. **for each** $H \in \mathrm{Sep}(J)$ **do:**
4. $\quad\quad \mathrm{OPT}[\mathrm{left}(J,H)] \coloneqq \mathrm{Schedule}\big(\mathrm{left}(J,H)\big)$
5. $\quad\quad \mathrm{OPT}[\mathrm{right}(J,H)] \coloneqq \mathrm{Schedule}(\mathrm{right}(J,H))$
6. $\mathrm{OPT}[J] := \min_{H \in \mathrm{Sep}(J)} \{\mathrm{OPT}[\mathrm{left}(J,H)] + \mathrm{OPT}[\mathrm{right}(J,H)] + 1\}$
7. $\mathrm{LUT}[\mathrm{core}(J), \#\mathrm{iso}(J)] = \mathrm{OPT}[J]$
8. Return $\mathrm{OPT}[J]$

How does this help?

Feasible Job sets

Let J be a *feasible set of jobs*.



time

	1	2	3	4
1	a	c	f	i
2	b	d	g	j
3		e	h	

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Let J be a *feasible set of jobs*.

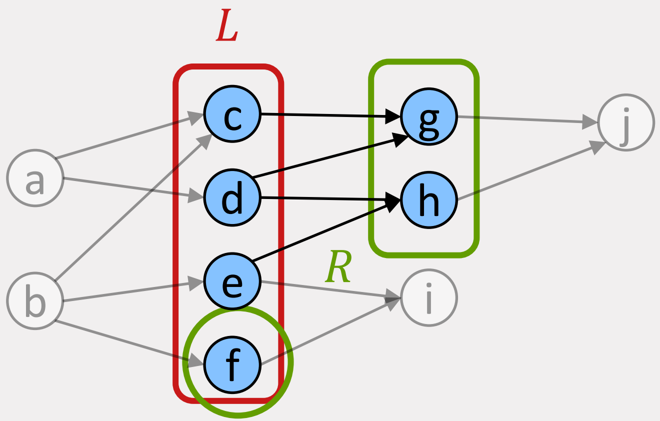
Jobs J can be described as

$$J = \mathrm{succ}[L] \cap \mathrm{pred}[R]$$

where

$L =$ minimal elements = sources of J

$R =$ maximal elements = sinks of J



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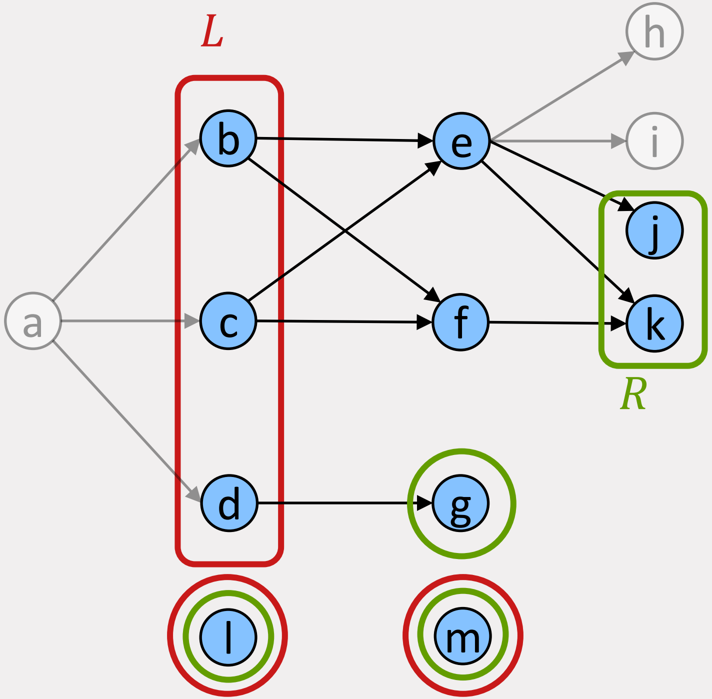

Feasible Job sets

Let J be a *feasible set of jobs*.

~~Jobs J~~ $\mathbf{core}(J)$ can be described as

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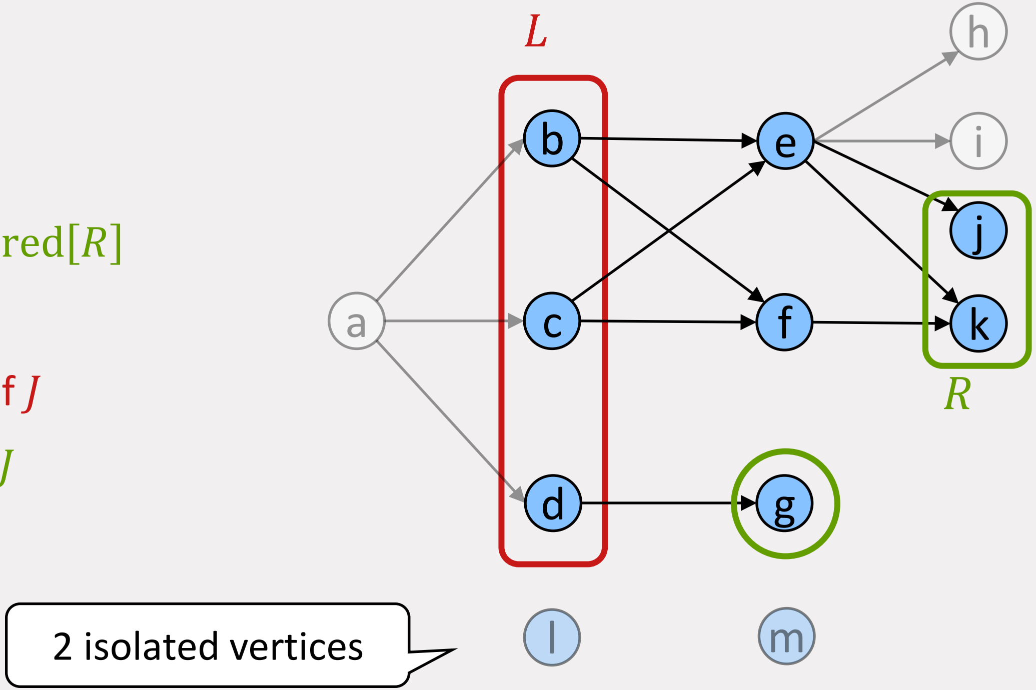

Feasible Job sets

Let J be a *feasible set of jobs*.

~~Jobs J~~ can be described as
core(J)

$$\mathbf{succ}(L)$$
$$\not{J} = \sout{\text{succ}[L]} \ \sout{\cap}\ \text{pred}[R]$$
$$\mathbf{core}(J)$$

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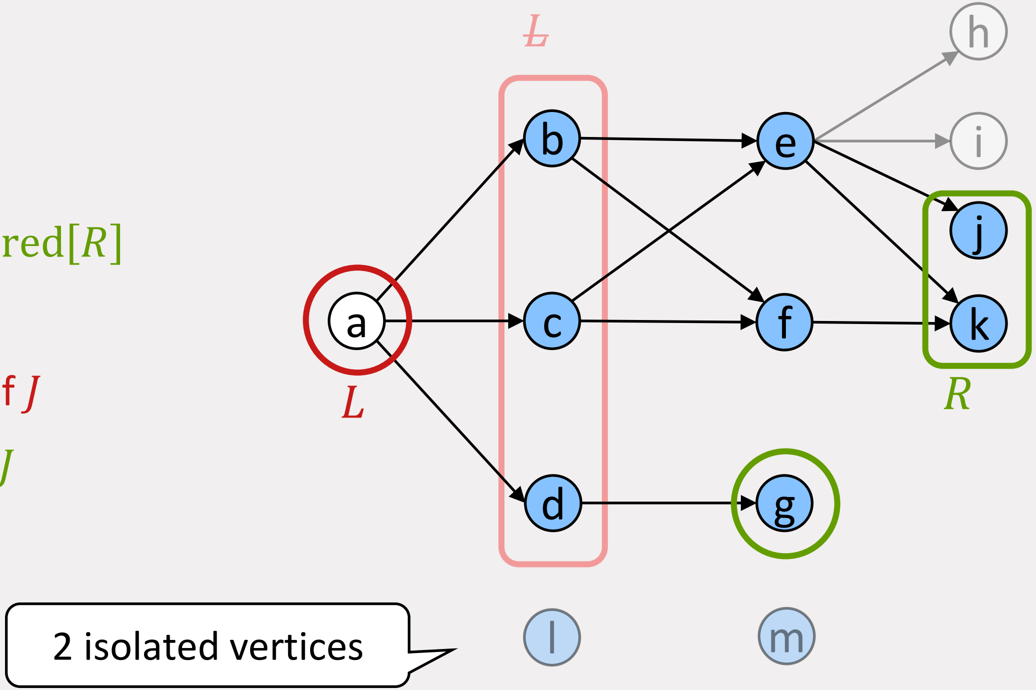
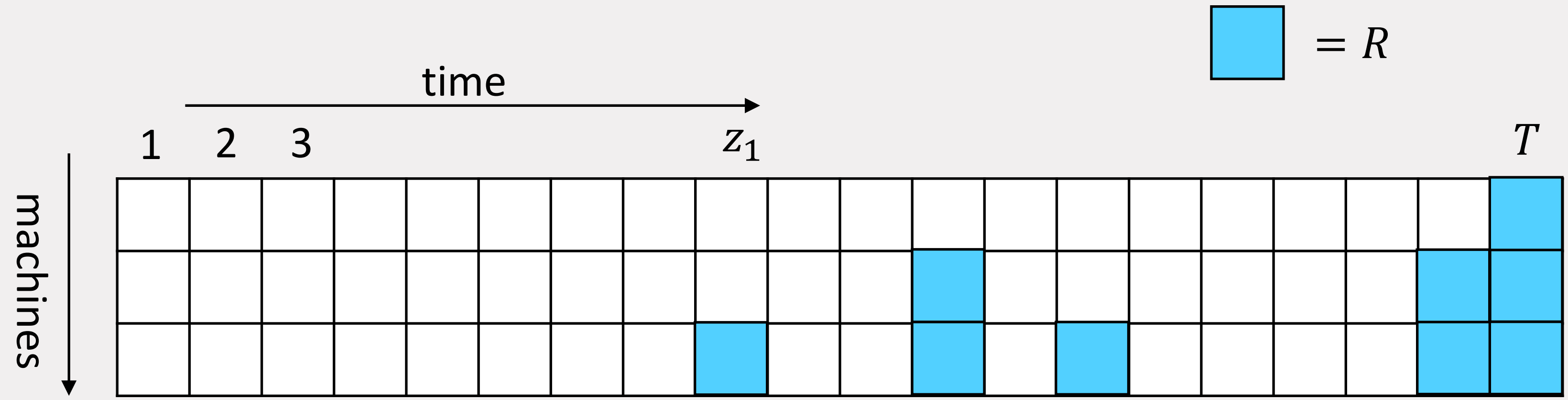


Going to the right

Assumption: $n = 3 \cdot T$



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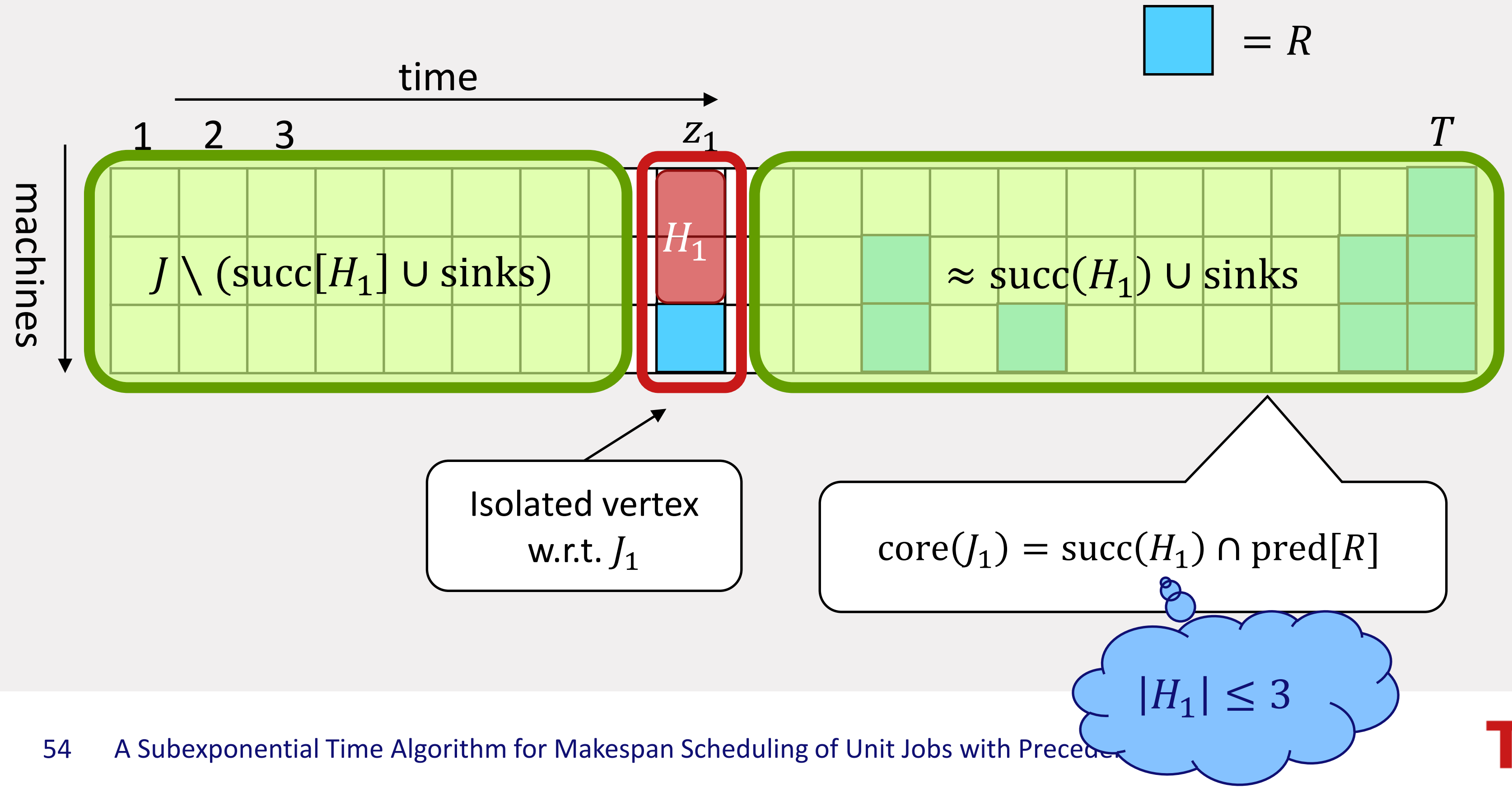

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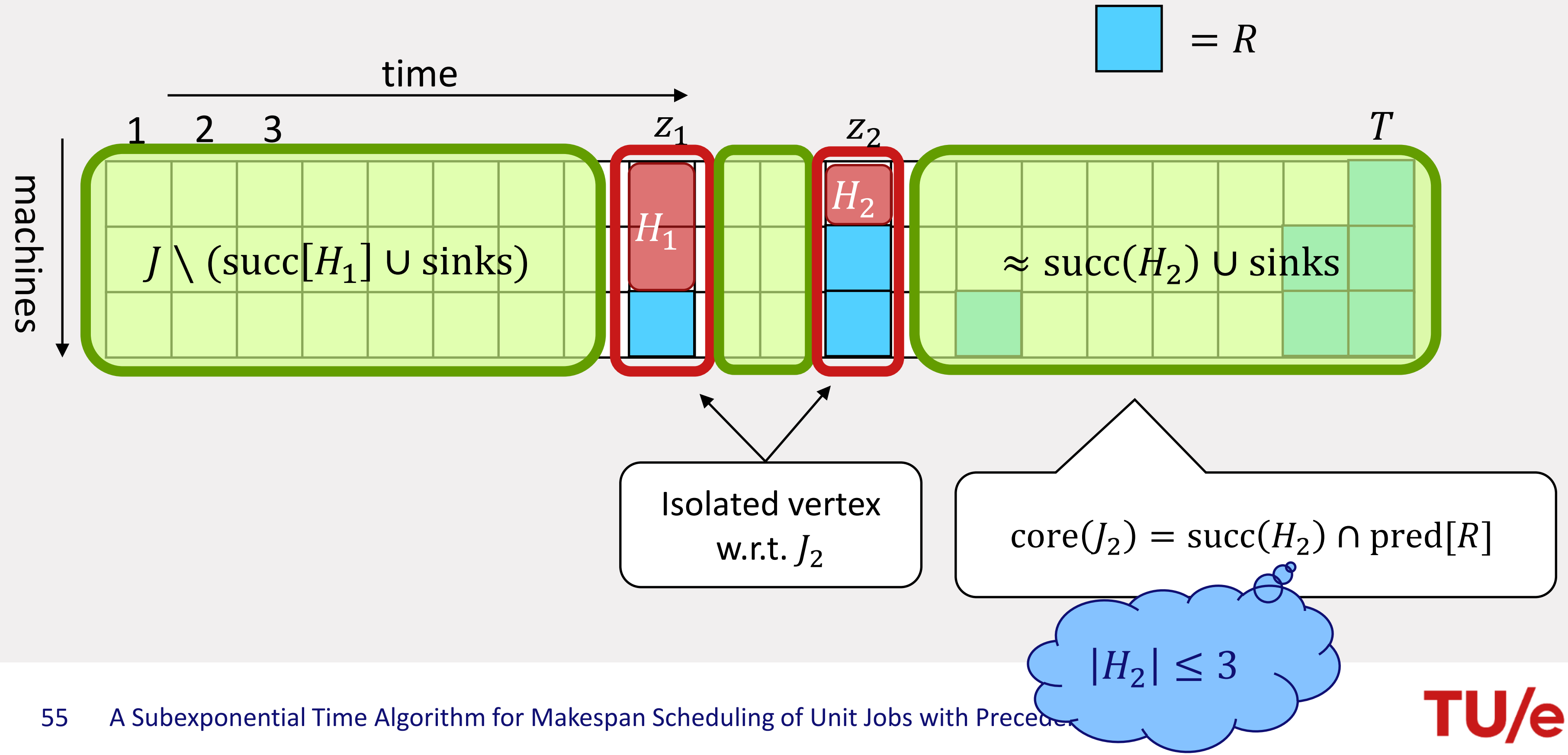
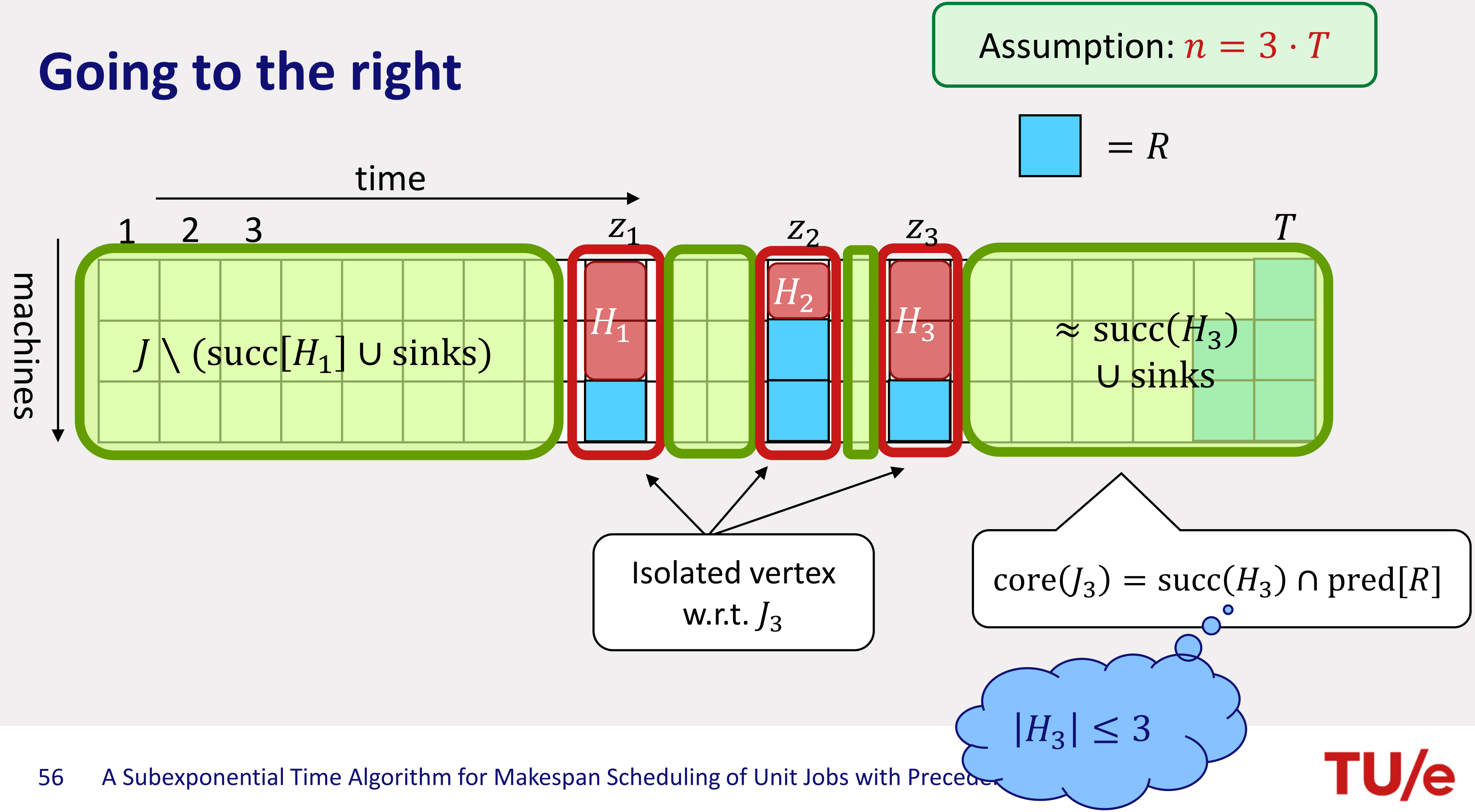


TU/e

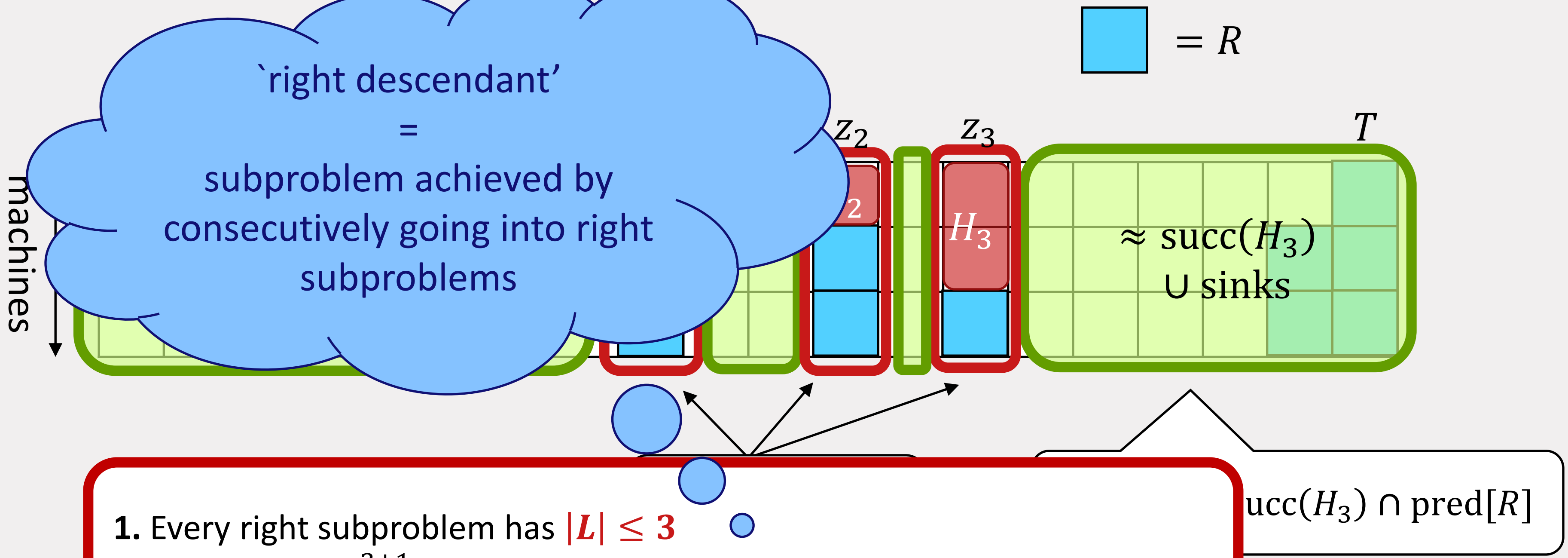
Going to the right



Going to the right

Assumption: $n = 3 \cdot T$

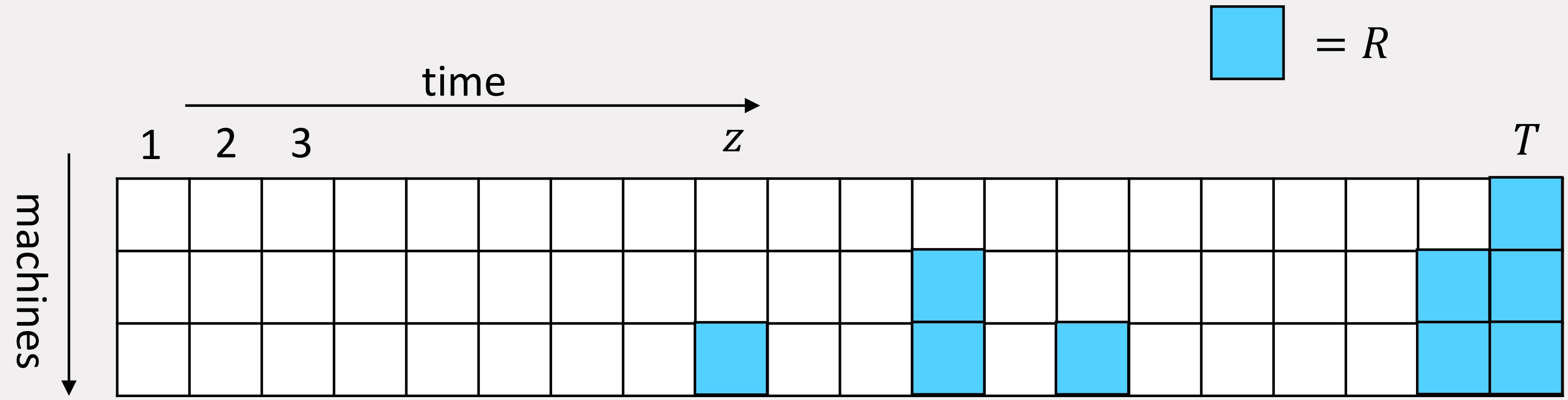
`right descendant'
=
subproblem achieved by consecutively going into right subproblems



1. Every right subproblem has $|L| \leq 3$
2. There are $\leq n^{3+1}$ different **`right descendants'**

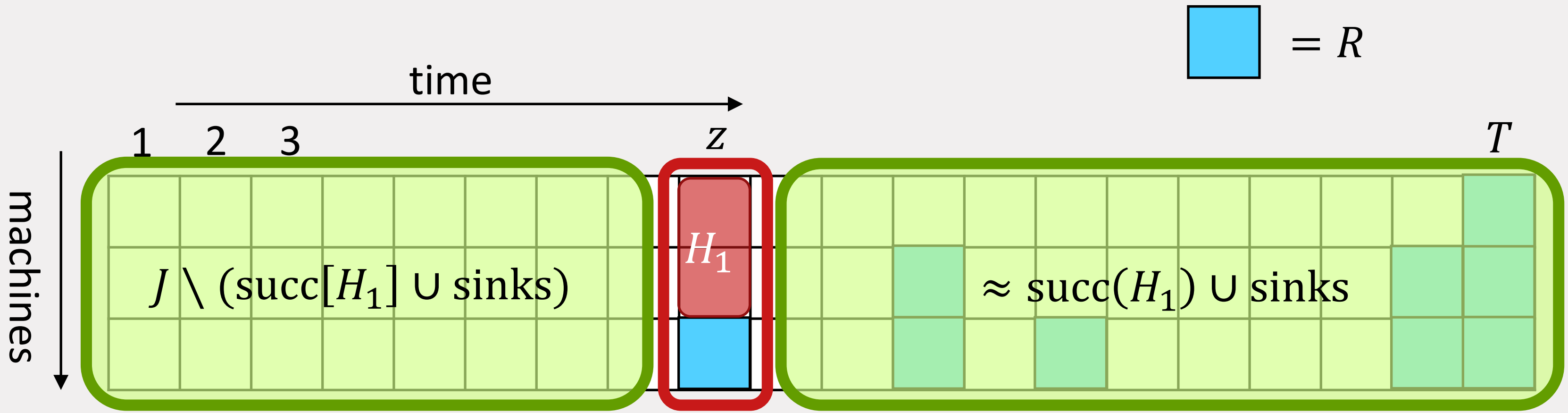
Going to the left

Assumption: $n = 3 \cdot T$



Going to the left

Assumption: $n = 3 \cdot T$



$\mathrm{core} = \mathbf{succ(L)} \cap \mathrm{pred}[R_{\mathrm{new}}]$

1. Every left subproblem has $|L| \le 3$
2. Problem size decreases by $|R|$

TU/e

Win-Win strategy

Invariant: $J = (\mathrm{succ}(L) \cap \mathrm{pred}[R]) \cup k$ isolated vertices, $|L| \leq 3$

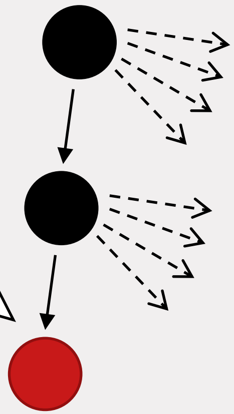
$\leq n^3$

Win-win strategy

$\leq n$

Case $|R| \leq \sqrt{n}$	Case $|R| > \sqrt{n}$
$\Rightarrow$ only $\binom{n}{\sqrt{n}} = 2^{O(\sqrt{n} \cdot \log n)}$ different R's	In next left step: make $\sqrt{n}$ jobs progress!

Branching Tree



Either already in Lookup Table:
- Base case

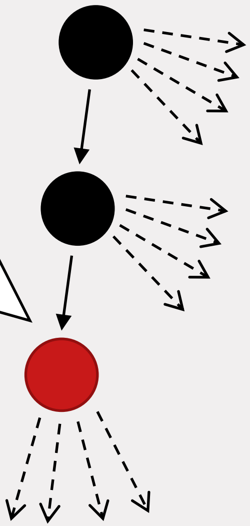
$= |R| \leq \sqrt{n}$

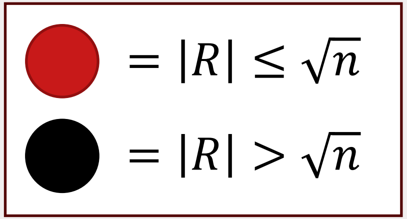
$= |R| > \sqrt{n}$

Branching Tree

Either already in Lookup Table:
- Base case

Or not yet in Lookup Table:

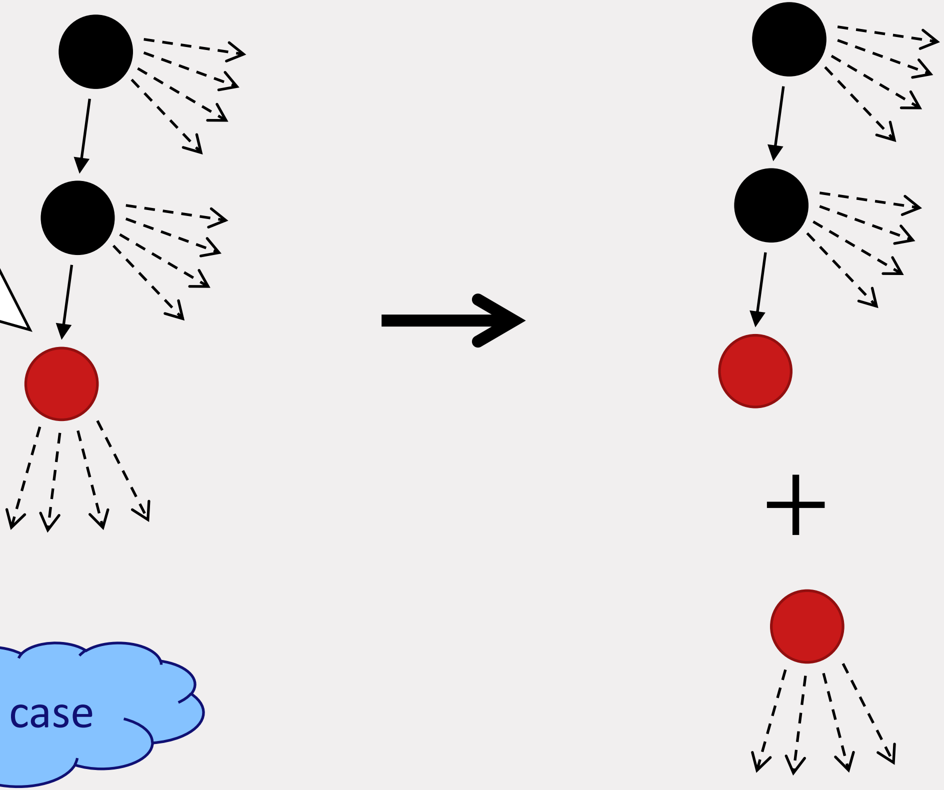


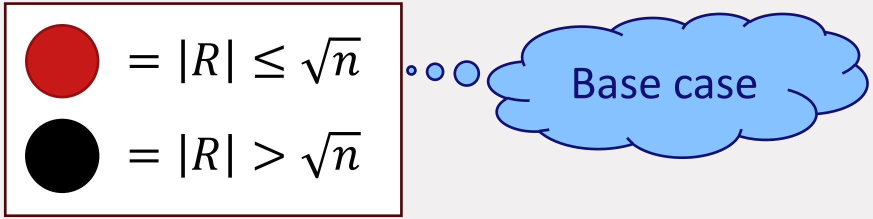


Branching Tree

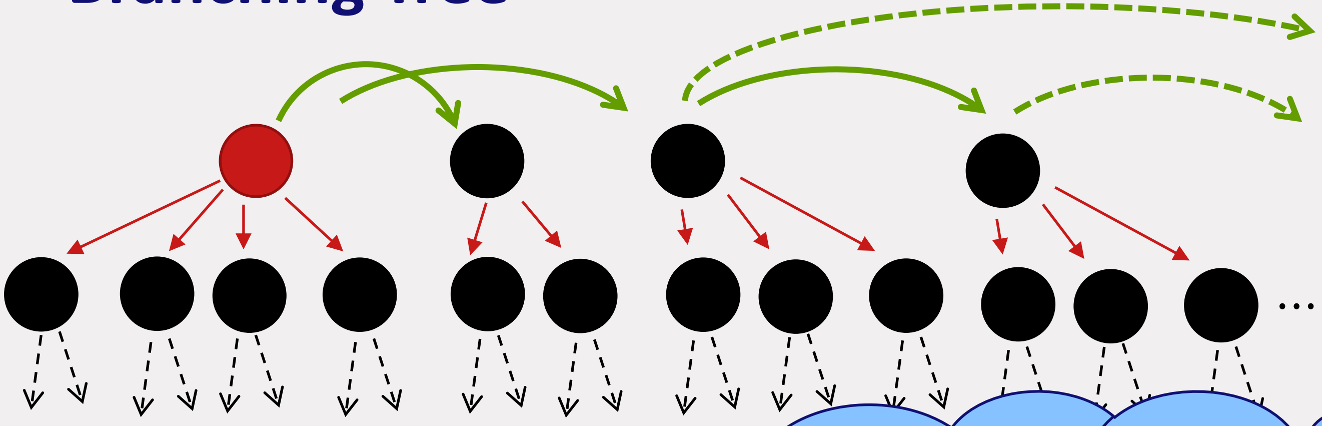
Either already in Lookup Table:
- Base case

Or not yet in Lookup Table:
- View as its `own tree’
- $\Rightarrow n^{O(\sqrt{n})}$ such trees





Branching Tree



$\leq n^4$ right descendants

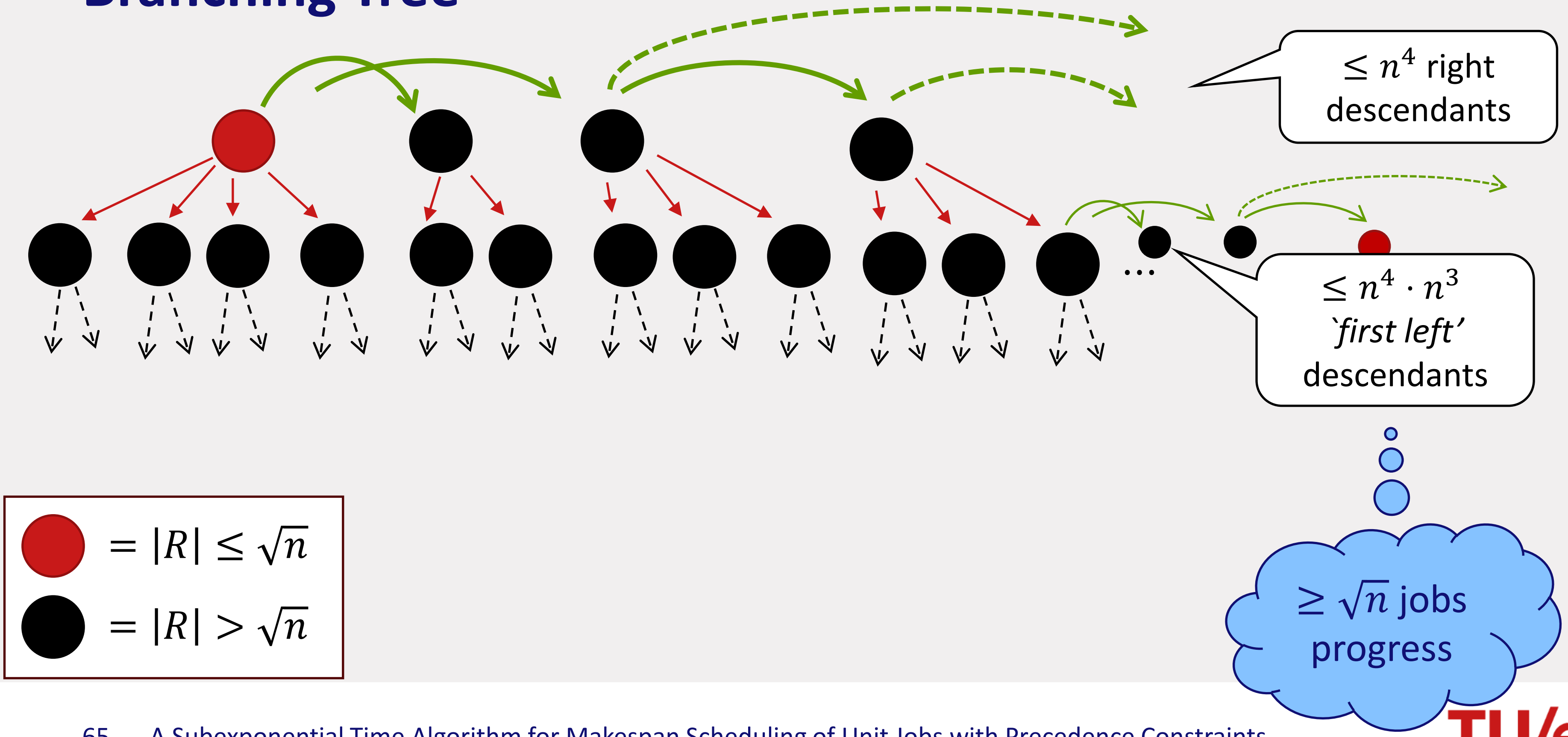
$\leq n^4 \cdot n^3$ `*first left*' descendants

`first left descendant'
=
subproblem achieved by consecutively going into right subproblems, then **once left**

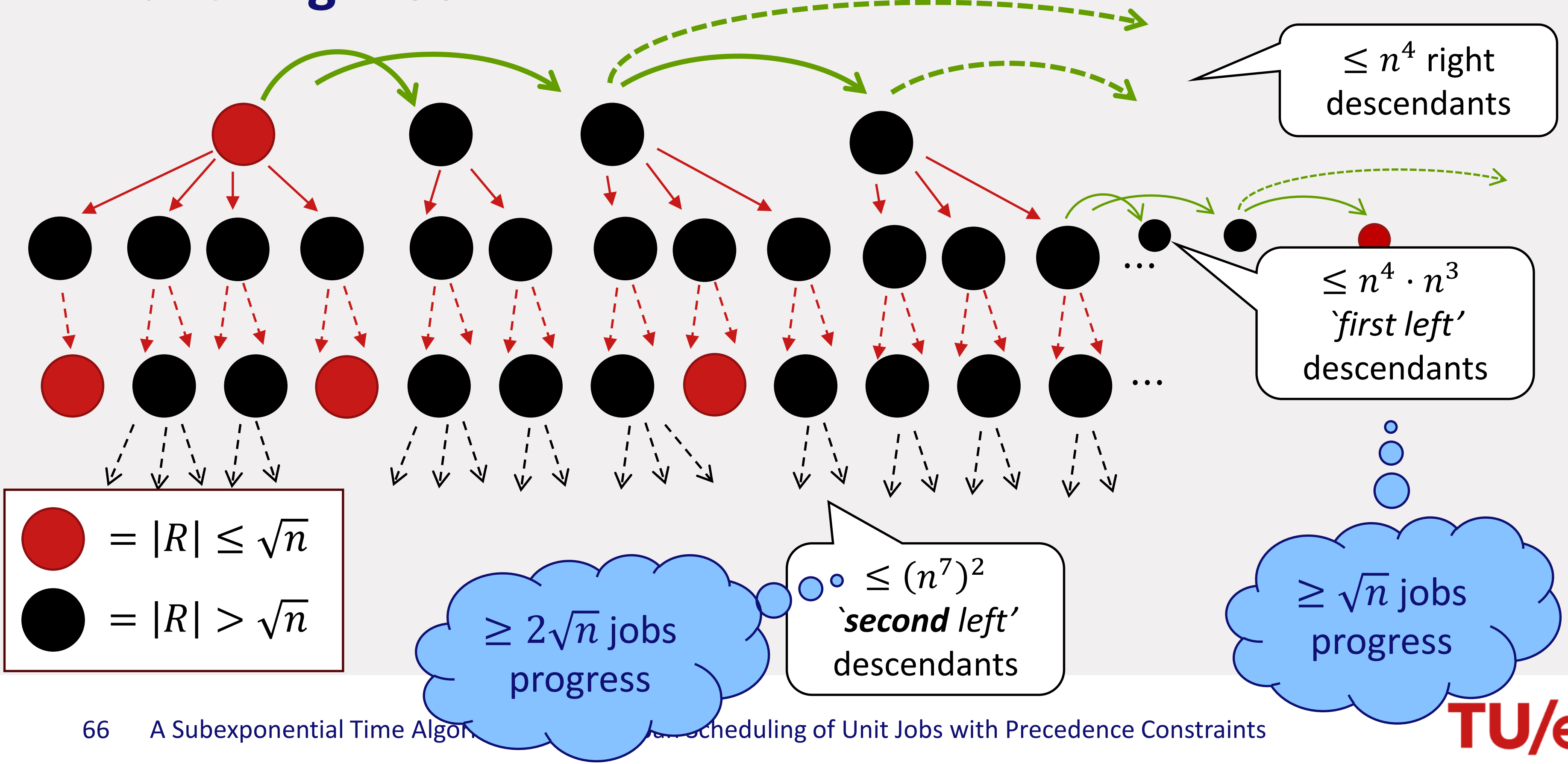
$\geq \sqrt{n}$ jobs progress

red circle	$= \lvert R \rvert \leq \sqrt{n}$
black circle	$= \lvert R \rvert > \sqrt{n}$

Branching Tree



Branching Tree



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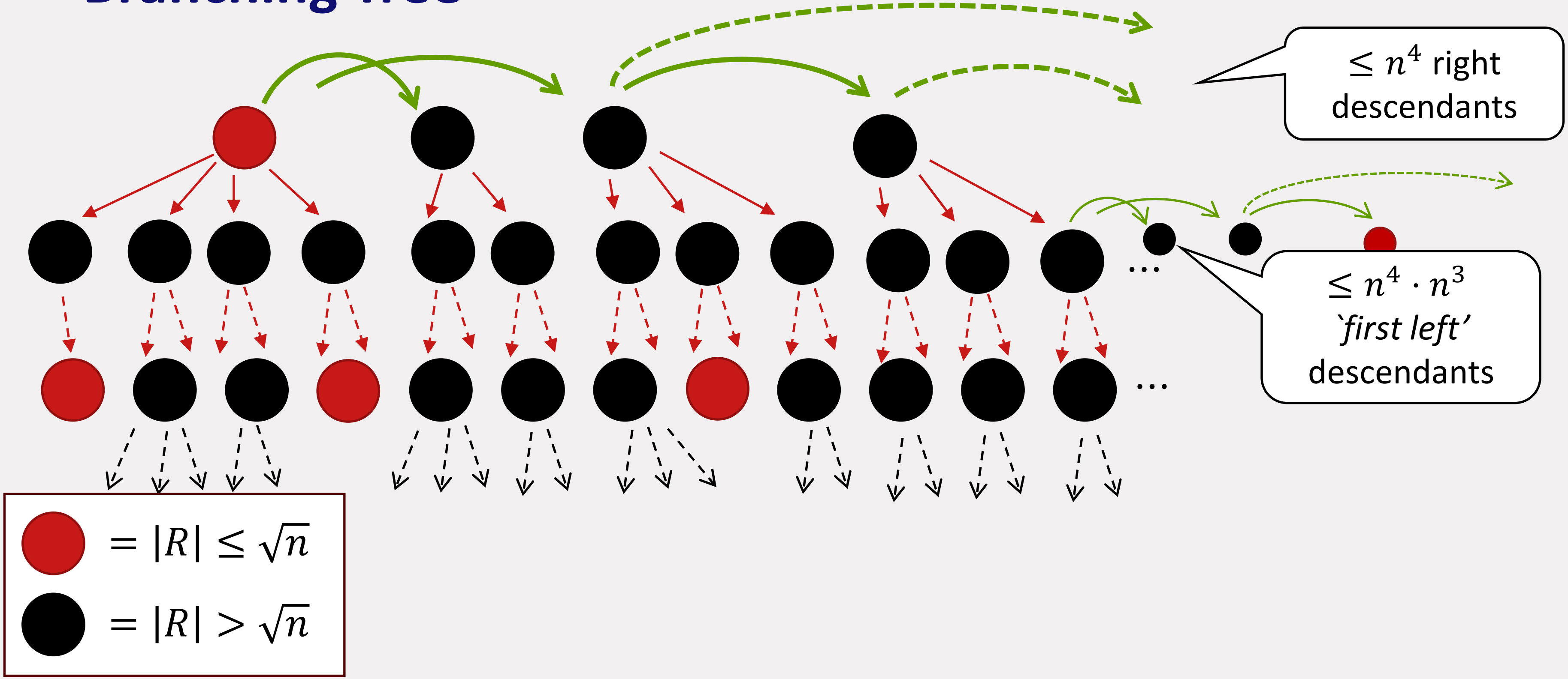

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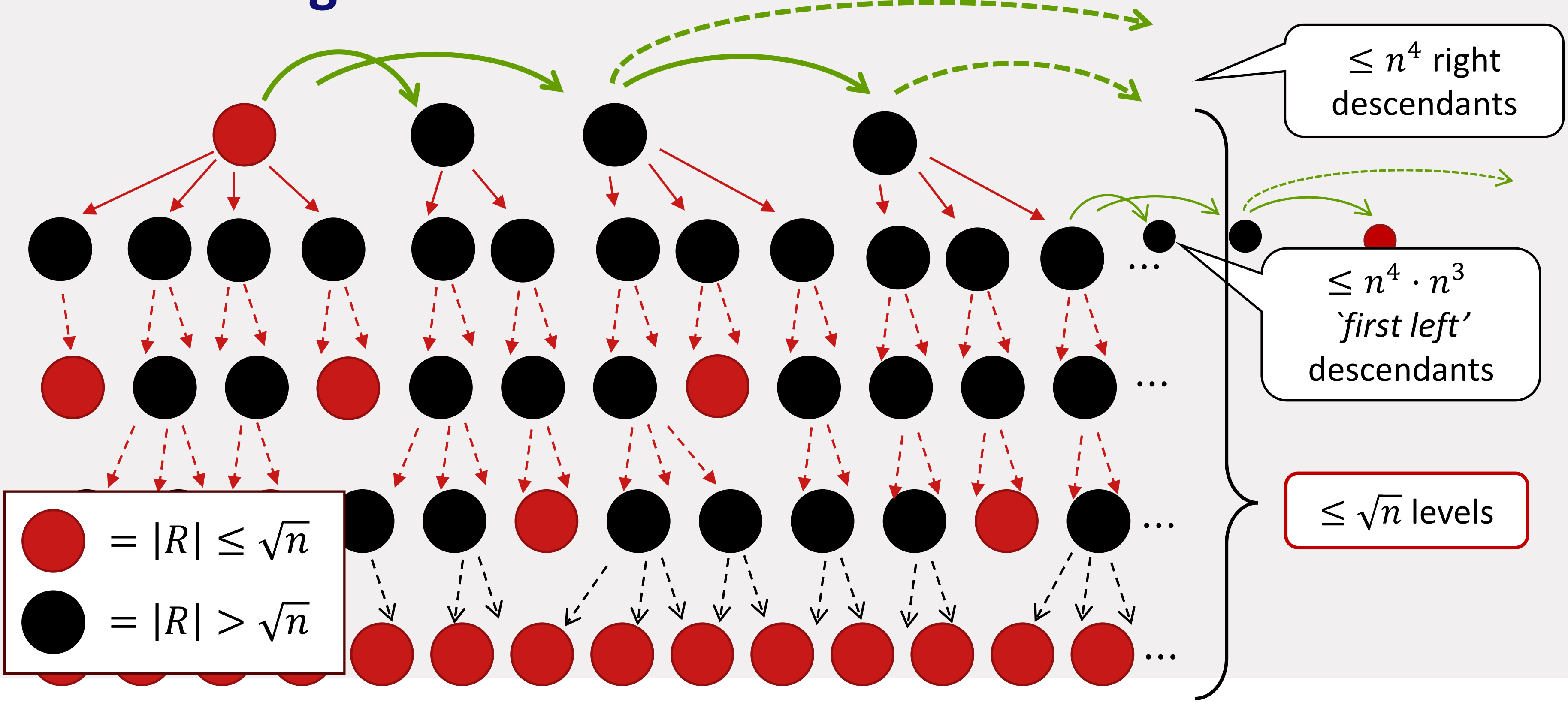

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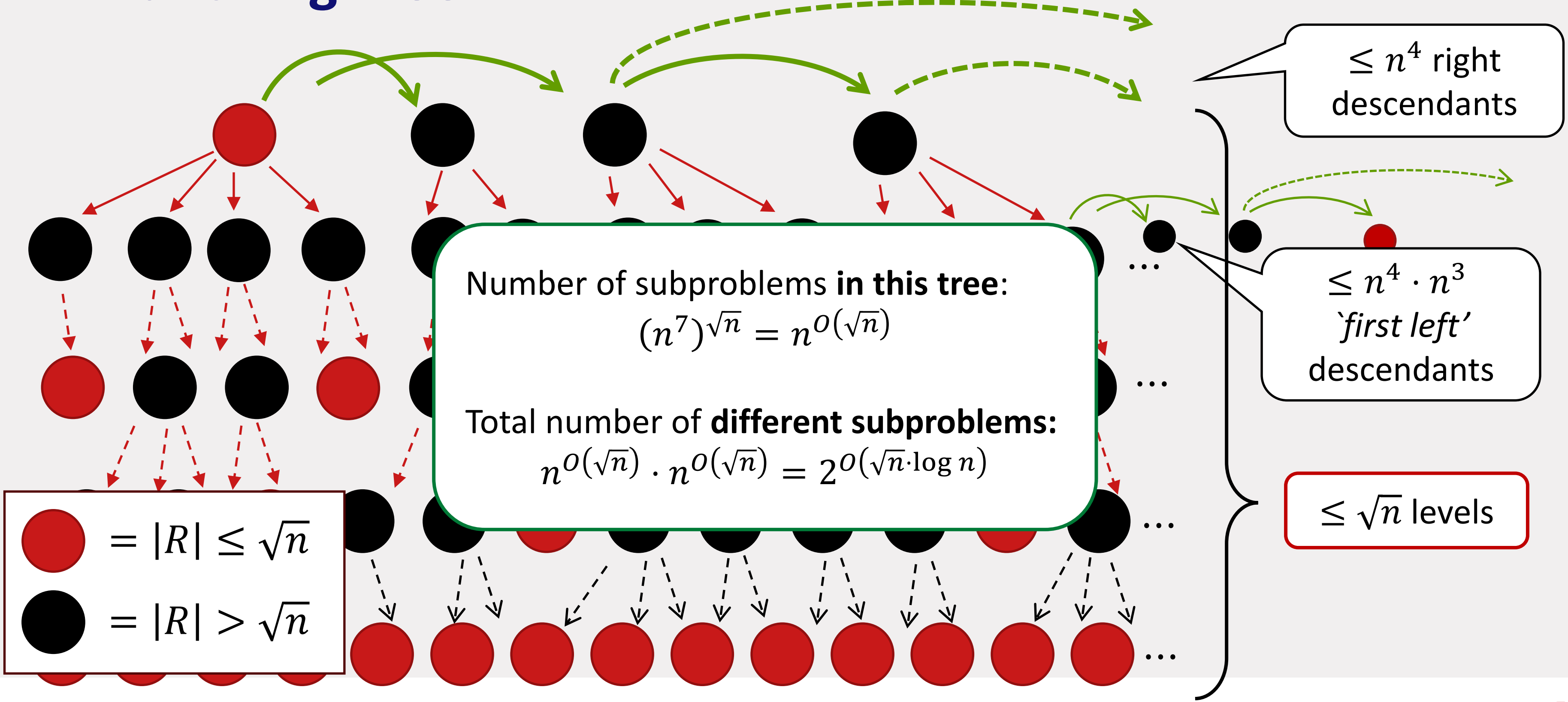

Number of subproblems **in this tree**:

$$(n^7)^{\sqrt{n}} = n^{O(\sqrt{n})}$$

Total number of **different subproblems:**

$$n^{O(\sqrt{n})} \cdot n^{O(\sqrt{n})} = 2^{O(\sqrt{n} \cdot \log n)}$$

Algorithm

Schedule(J):

1. **return** $\mathrm{LUT}[\mathrm{core}(J), \#\mathrm{iso}(J)]$ if it was already set
2. **if** $J = \emptyset$ **return** 0
3. **for each** $H \in \mathrm{Sep}(J)$ **do:**
4. $\quad\mathrm{OPT}[\mathrm{left}(J,H)] \coloneqq \mathrm{Schedule}\big(\mathrm{left}(J,H)\big)$
5. $\quad\mathrm{OPT}[\mathrm{right}(J,H)] \coloneqq \mathrm{Schedule}(\mathrm{right}(J,H))$
6. $\mathrm{OPT}[J] := \min_{H \in \mathrm{Sep}(J)} \{\mathrm{OPT}[\mathrm{left}(J,H)] + \mathrm{OPT}[\mathrm{right}(J,H)] + 1\}$
7. $\mathrm{LUT}[\mathrm{core}(J), \#\mathrm{iso}(J)] = \mathrm{OPT}[J]$
8. Return $\mathrm{OPT}[J]$

$\mathrm{Sep}(J) \coloneqq \{ H \subseteq J$ s.t.
(1) $|H| \leq 3$,
(2) H is antichain,
(3) $|H \setminus \mathrm{sinks}(J)| < 3\}$

$\mathrm{left}(J,H) \coloneqq J \setminus \big(\mathrm{succ}[\mathrm{H}] \cup \mathrm{sinks}(J)\big)$
$\mathrm{right}(J,H) \coloneqq J \cap \big((\mathrm{succ}(H) \cup \mathrm{sinks}(J)\big) \setminus H$

Only $2^{O(\sqrt{n} \cdot \log n)}$ different problems encountered

Corollaries

Our result:

$Pm|prec, p_j = 1|C_{\max}$ can be solved in $\left(1 + \frac{n}{m}\right)^{O(\sqrt{nm})}$ time.

Corollary 1

$Pm|prec, p_j = 1|C_{\max}$ can be solved in subexponential time whenever $m = o(n)$.

Corollary 2

$\boldsymbol{P}|prec, p_j = 1|C_{\max}$ can be solved in $1.997^n \cdot poly(n)$ time.

Conclusion

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Main result:

$P3|prec, p_j = 1|C_{\max}$ in $2^{O(\sqrt{n} \cdot \log n)}$ time.

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$P3|prec, p_j = 1|C_{\max}$ in $2^{O(\sqrt{n}\cdot\log n)}$ time.

Key idea's:

1. Use of look-up table
2. Keeping track of core + # isolated vertices
3. Finding win-win strategy using number of sinks

Open Problems:

- $P3|prec, p_j = 1|C_{\max}$ in quasi-polynomial time?

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- Approximation algorithms, does a PTAS exist (for fixed m)?
  - QPTAS by
    - Garg 2018
    - Li, 2021
    - Das, Wiese, 2022

$(1+\varepsilon)$-approximation in $n^{O\left(\frac{m^4}{\varepsilon^3}\log^3 \log n\right)}$ time

Open Problems:

Thanks for your attention!

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